Published by Institute of Physics Publishing for SISSA/ISAS
Received: *May 22, 2003*
Accepted: *July 21, 2003*

Vector operators in the BMN correspondence

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Abstract: We consider a BMN operator with one scalar, ϕ and one vector, $D_\mu Z$, impurity field and compute the anomalous dimension both at planar and torus levels. This *mixed* operator corresponds to a string state with two creation operators which belong to *different* SO(4) sectors of the background. The anomalous dimension at both levels is found to be the same as the scalar impurity BMN operator. At planar level this constitutes a consistency check of BMN conjecture. Agreement at the torus level can be explained by an argument using supersymmetry and supression in the BMN limit. The same argument implies that a class of fermionic BMN operators also have the same planar and torus level anomalous dimensions. Implications of the results for the map from $\mathcal{N} = 4$ SYM theory to string theory in the pp-wave background are discussed.

Keywords: Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

AdS/CFT correspondence, as an explicit realization of string/gauge duality, passed many tests performed in the supergravity approximation over the last four years. Yet, this correspondence suffers, at least quantitatively, from the obstacles in extending it into a full string theory/gauge theory duality. This is mainly due to the lack of a clear dictionary between massive string modes of IIB on $AdS_5 \times S^5$ and gauge invariant operators in the dual $\mathcal{N} = 4$ SYM at strong coupling. Specifically, the massive modes are dual to operators in long multiplets of SYM and have divergent anomalous dimensions as $\lambda = g^2_{\rm YM} N \to \infty$. This fact, among others, hinders our understanding of strongly coupled gauge theory as a string theory.

However, Berenstein, Maldacena and Nastase has taken an important step in this direction [1]. BMN focused on a particular sector of the Hilbert space of gauge theory in which the observables themselves also scale with λ, such that they remain nearly BPS, namely their anomalous dimension acquire only finite corrections. They identified the operators which carry large R-charge, J, under a U(1) subgroup of SU(4) — the full R-symmetry of $\mathcal{N}=4$ SYM — and this R-charge is subject to a scaling law as, $J \approx \sqrt{N}$. As described in the next section in detail, these are essentially single trace operators that involve a chain of J fields which are $+1$ charged under U(1) with a few U(1)-neutral *impurity* fields inserted in the chain and the number of these *impurities* corresponds to the number of string excitations on the world-sheet.

The conjecture is that, BMN operators of SYM are in one-to-one correspondence with the string states which carry large angular momentum, J, along the equator of S^5. The systematic way of taking this particular limit in the gravity side is to consider a null geodesic along the equator and blow up the neighborhood of the geodesic through constant scaling of the metric [2, 3]. The homogeneity property of Einstein-Hilbert action guarantees that end-product is also a solution of the Einstein equations, and in fact it is a plane-wave geometry supported by the RR 5-form,

$$ds^2 = -4dx^+dx^- - \mu^2 z^2 dx^{+2} + dz^2\,, \qquad F_{+1234} = F_{+5678} = \frac{\mu}{4\pi^3 g_{\rm s}\alpha'^2}\,, \tag{1.1}$$

where z^i span the 8 dimensional transverse space. This solution preserves 32 supercharges, just as $AdS_5 \times S^5$. It is a particular example of the Penrose limit, [4, 5] which generally shows that any space-time in general relativity yield plane-wave geometry as a limit. What makes the background (1.1) very attractive for string theory is that quantization of string theory in pp-wave background is known [6]. RR 5-form field strength curves the space-time in such a way that oscillator modes of the 8 transverse world sheet fields (and their fermionic partners) in the light-cone gauge acquire a mass proportional to F. In turn light-cone energy of string modes read,

$$p^- = \mu \sum_{n=-\infty}^{\infty} N_n \sqrt{1 + \frac{n^2}{(\mu p^+ \alpha')^2}} \tag{1.2}$$

where N_n is the occupation number of n-th oscillator mode. In BMN dictionary this energy is dual to $\Delta - J$ of the corresponding BMN operator in SYM, whereas the light-cone momentum p^+ is proportional to the R-charge, J. In detail, the correspondence is,

$$\mu p^+ \alpha' = \frac{J}{\sqrt{\lambda}}\,, \qquad \frac{2p^-}{\mu} = \Delta - J\,, \qquad g_{\rm YM}^2 = 4\pi g_{\rm s}\,. \tag{1.3}$$

Utilizing the AdS/CFT correspondence BMN found a relation between the anomalous dimensions of the BMN operators and the oscillation number of the corresponding string states,

$$(\Delta - J)_{n,\,N_n} = N_n \sqrt{1 + \frac{g_{\rm YM}^2 N n^2}{J^2}}\,. \tag{1.4}$$

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We see that in the large-N limit, only the operators whose R-charge scale as $J \approx \sqrt{N}$ stays in the spectrum (along with chiral primaries) as the other observables decouple. Therefore the BMN limit in detail is,

$$N \to \infty, \text{ with } \frac{J}{\sqrt{N}} \text{ and } g_{\mathrm{YM}} \text{ fixed.} \tag{1.5}$$

This limit differs from the usual large-N limit of gauge theory in that the observables are also scaled as J is not fixed. Therefore neither $\lambda \to \infty$ implies infinite coupling in SYM nor $1/N \to 0$ implies planarity. In fact, a detailed study of free and coupled correlation functions in the BMN sector of SYM revealed that [7, 8, 9] theory develops a different effective coupling constant,

$$\lambda' = \frac{g_{\mathrm{YM}}^2 N}{J^2} = \frac{1}{(\mu p^+ \alpha')^2}\,, \tag{1.6}$$

and a different genus expansion parameter,

$$g_2^2 = \left(\frac{J^2}{N}\right)^2 = 16\pi^2 g_{\mathrm{s}}^2 (\mu p^+ \alpha')^4\,. \tag{1.7}$$

As a result, in the modified large-N limit, (1.5), one has an interacting gauge theory with a tunable coupling constant λ'. However non-planar diagrams are not ignorable necessarily. A direct consequence of this non-planarity in SYM interactions can be observed as mass renormalization of string states [9]. In [9], $\mathcal{O}(\lambda')$ contribution to the string state mass was related to torus level contribution to $\Delta - J$ and this value was computed. They observed that the effective string coupling (which appears in the physical quantities like $\Delta - J$) is not identical to the genus counting parameter g_2 but modified with $\mathcal{O}(\lambda')$ SYM interactions as[1]

$$g_s' = g_2 \sqrt{\lambda'}\,. \tag{1.8}$$

Now, we observe a very significant fact about the BMN limit. Since λ' and g_s' are independent and both can be made arbitrarily small, in that regime one has *a duality between weakly coupled gauge theory and interacting perturbative string theory.* This provides not only a duality between observables in SYM and string states on pp-wave background but also an explicit map between gauge and string interactions.

This interacting level string/gauge duality was investigated in detail by the authors of [9]. They proposed a relation between the three-string vertex and the three-point function in the gauge theory. At weak coupling in SYM (small λ') the formula relates the matrix element of string field theory light-cone hamiltonian between one-string and two-string states to the coefficient of $\mathcal{O}(\lambda')$ three-point function of corresponding BMN operators;

$$\langle i'|P^-|j'\rangle|k'\rangle = \mu(\Delta_i - \Delta_j - \Delta_k) C_{ijk} \tag{1.9}$$

where $|i'\rangle$ are free string states with the normalization $\langle i'|j'\rangle = \delta_{ij}$ and C_{ijk} is the coefficient of free planar three-point function of corresponding operators. This relation was recently

[1] In [10], a generalization of g_s' to arbitrary values of λ' was proposed.

argued to be correct in two different approaches. In the light-cone string field theory [11] approach, one considers the cubic interaction term which has two constituents: a delta-functional overlap which is required by the continuity of world-sheet fields and a *prefactor* which acts on this delta-functional whose presence is required by supersymmetry. By considering the leading order corrections in $1/\mu$ to the delta-functional, authors of [12, 13] found agreement with (1.9) in some special cases.[2] Finally, Spradlin and Volovich [15] obtained perfect agreement with (1.9) by taking $1/\mu \to 0$ limit of the *prefactor*. In a totally independent approach, Verlinde [16] developed a string bit formalism in terms of supersymmetric quantum mechanics for which the basic interaction also agrees with (1.9).[3]

Until now all of the calculations in the literature involved BMN operators with two scalar type impurities. This operator corresponds to two-string excitations in the transverse directions 1,2,3 or 4 and the form (1.9) was proposed only for scalar impurity BMN operators. However, one can generally construct BMN operators also with *vector* or *fermionic* type impurities. This manuscript is devoted to a study of the *vector* type BMN operator which is formed with one scalar type impurity (a string excitation in 1,2,3,4) and one vector type impurity (a string excitation in 5,6,7,8). We study the interacting two-point function of the vector operator both at planar and genus one levels and compute the *planar* and *torus* level anomalous dimensions in $\mathcal{O}(\lambda')$. The torus dimension gives the first order, $\mathcal{O}(g_s'^2)$, mass renormalization of the corresponding string state. Our main result is that *both* planar *and* torus anomalous dimensions of the vector operator are the same as the anomalous dimensions of the scalar BMN operator.

The tensor structure of correlation functions of vector BMN operators is greatly restricted by conformal symmetry. In particular the unique space-time form of two-point functions is proportional to $J_{\mu\nu}$, the jacobian of conformal inversion. Furthermore the vector impurity is in form of the covariant derivative $D_\mu Z$, therefore the restrictions of gauge invariance on the correlators will be more apparent. Although the calculations are non-trivial even at the planar level we find that the forms required by these symmetry principles do emerge after combining several contributions. This fact improves our confidence in the calculations.

Our motivation in studying this specific operator is two-fold. One technical motivation is to explore how the restrictions due to symmetry principles mentioned above are implemented. Secondly, our torus level result allows us to explore the implications for the interacting level string/gauge duality in the particular case of vector operators.

The equality of planar level scalar and vector anomalous dimensions, $(\Delta - J)_{\text{planar}}$, is required by the consistency of BMN conjecture since (1.4) is independent of the transverse space index $i = 1, \ldots, 8$. Therefore our planar level result provides a non-trivial consistency check on the conjecture. However the equality at the torus level comes as a surprise at first and makes us suspect that there exist a superconformal transformation which relates the scalar and vector operators hence equates the anomalous dimensions at all genii and all orders in λ'. In section 6 we indeed find a two-step SUSY transformation which maps

[2] See also [14] for a test of this conjecture on supergravity modes.

[3] Later developments have revealed that eq. (1.9) holds only when the RHS is given in a different basis (called "string basis"), see ref [27, 28].

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the scalar operator onto the vector one plus some correction terms. We show that these corrections terms yield an $\mathcal{O}(1/J)$ modification both for the planar and torus anomalous dimensions. Hence in the BMN limit these corrections are negligible and one arrives at another proof for the equality of scalar and vector anomalous dimensions. However, the fact that this SUSY map is non-exact but $1/J$ corrected prevents us from making a more general conclusion about the equality the dimensions at higher genii or higher orders in λ'. As a bonus, we also show that the fermionic type BMN operators which show up in the mentioned SUSY transformation possess the same planar and torus anomalous dimensions as the scalar operator.

Our torus level result also has implications for the map between three-point functions in field theory and the cubic string vertex for the states with one scalar ($i = 1, \dots, 4$) and one vector ($i = 5, \dots, 8$) mode excited. The aforementioned *prefactor* of string field theory weighs string interactions in the $i = 1, \dots, 4$ and $i = 5, \dots, 8$ directions with opposite sign, so r.h.s. of (1.9) actually vanishes in this case [15]. A unitarity sum of the type performed for scalar excitations in [9] would predict vanishing torus level anomalous dimensions in contradiction to our field theory result. This indicates that the anomalous dimension must come from another place in the string calculation, perhaps from a *contact term*. These issues will be discussed again in the last section of the paper.

The paper is organized as follows. In the next section we review the map between string states and BMN operators and introduce a neat "q-variation" notation to handle string excitations. Then we give the definition of the vector operator and compute free two-point functions at planar level. In section 3 we compute free two-point function of the vector operator at the torus level. This will provide a warm-up exercise for our interacting torus level calculations. Section 4 presents the calculation of planar anomalous dimension and develops necessary techniques to be used also at torus level. In section 5 we calculate the torus dimension of the vector operator. Section 6 presents a SUSY argument to understand why one obtains same anomalous dimension for vector and scalar operators at both planar and torus levels. In section 7 we conclude with a summary of our results and discuss possible resolutions of the aforementioned contradiction between string field theory and gauge theory results for the torus dimension of the vector operator. Appendices fill in some of the details of the calculations.

2. State/operator map

2.1 BMN operators

In this section we shall first review the BMN state/operator map between first few bosonic excitations of IIB string on the pp-wave background and corresponding operators in SYM on $\mathbf{R} \times S^3$. First of all, string vacuum corresponds to the BPS operator (with appropriate normalization),[4]

$$O^J = \frac{1}{\sqrt{J(N/2)^J}} \operatorname{Tr}(Z^J) \longleftrightarrow |0, p^+\rangle. \tag{2.1}$$

[4]We use the common convention $\operatorname{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

To discuss the excitations it is convenient to form complex combinations of the 6 scalars of SYM as,

$$Z^1 = Z = \frac{X^5 + iX^6}{\sqrt{2}}\,, \qquad Z^2 = \phi = \frac{X^1 + iX^2}{\sqrt{2}}\,, \qquad Z^3 = \psi = \frac{X^3 + iX^4}{\sqrt{2}}\,. \tag{2.2}$$

Operators corresponding to the supergravity modes, i.e. world-sheet momentum $n = 0$, are obtained from O^{J+1} by the action of SO(6), conformal or SUSY lowering operations. For example, the particular SO(6) operation $\delta_\phi Z = \phi$ acting on O^{J+1} yields (by the cyclicity of trace),

$$O^J_\phi = \frac{1}{\sqrt{J}}\delta_\phi O^{J+1} = \frac{1}{\sqrt{(N/2)^{J+1}}}\,\mathrm{Tr}(\phi Z^J)\,. \tag{2.3}$$

This is in correspondence with the supergravity mode $\alpha_0^{\phi\,\dagger}|0,p^+\rangle$ where $\alpha_0^\phi = \frac{1}{\sqrt{2}}(\alpha^1 - i\alpha^2)$. Similarly $\delta_\psi Z = \psi$ and the translation D_μ yields other bosonic supergravity modes,

$$\begin{aligned} O^J_\psi &= \frac{1}{\sqrt{(N/2)^{J+1}}}\,\mathrm{Tr}(\psi Z^J) \longleftrightarrow \alpha_0^{\psi\,\dagger}|0,p^+\rangle \\ O^J_\mu &= \frac{1}{\sqrt{(N/2)^{J+1}}}\,\mathrm{Tr}(D_\mu Z Z^J) \longleftrightarrow \alpha_0^{\mu\,\dagger}|0,p^+\rangle \end{aligned} \tag{2.4}$$

where $\mu = 5, 6, 7, 8$ and $\alpha_0^\psi = \frac{1}{\sqrt{2}}(\alpha^3 - i\alpha^4)$. To find the operator dual to a supergravity state with N_0 excitations one simply acts on O^{J+N_0} with N_0 lowering operators.

Turning now to the string excitations $n \neq 0$, we see that momentum conservation on the world-sheet prohibits creation of a single-excitation state with nonzero n. Therefore the next non-trivial string state involves two creation operators. Corresponding nearly BPS operators are introduced in [1] and discussed in detail in the later literature but we would like to present here a slightly different approach with a more compact notation. This will prove very useful when we discuss interactions of BMN operators. To generalize from supergravity to string modes let us introduce a "quantized" version of the derivation rule and define a *q-variation* by

$$\begin{aligned} \delta^q(f_1(x)f_2(x)\ldots f_k(x)) \;&=\; \delta^q f_1(x) f_2(x)\ldots f_k(x) + q f_1(x)\delta^q f_2(x)\ldots f_k(x) + \\ &\quad + \cdots + q^{k-1} f_1(x)\ldots f_{k-1}(x)\delta^q f_k(x) \end{aligned} \tag{2.5}$$

where f_i are arbitrary operators and q is an arbitrary complex number to be determined below. With this notation, the operator dual to single-excitation state, say $\alpha_n^{\phi\,\dagger}|0,p^+\rangle$, can be obtained by acting on O^{J+1} with q-variation δ^q_ϕ with q depending on n. By cyclicity of trace one gets,

$$\frac{1}{\sqrt{J}}\delta^q_\phi O^{J+1} = \frac{1}{J\sqrt{(N/2)^J}}\left(\sum_{l=0}^{J} q^l\right)\mathrm{Tr}(\phi Z^J)\,.$$

As mentioned above, this should vanish by momentum conservation for $n \neq 0$ and should reduce to (2.3) for $n = 0$. This determines q at once,

$$q = e^{2\pi i n/(J+1)}\,.$$

Let us now determine the operator dual to the two-excitation state, $\alpha_m^{\psi\,\dagger}\alpha_n^{\phi\,\dagger}|0,p^+\rangle$. This is obtained by action of $\delta^{q_2}_\psi\delta^{q_1}_\phi$ on O^{J+2} with q_1 and q_2 depending on n, m respectively. A single q-variation should vanish as above, hence fixing $q_1 = e^{2\pi in/(J+2)}$, $q_2 = e^{2\pi im/(J+2)}$. Double q-variation does not vanish in general because q-variation do not commute with cyclicity of trace. Note further that the case where both δ_ϕ and δ_ψ acts on the same Z vanishes by the flavor algebra. Even if this term had not been zero, it could be neglegted in the "dilute gas" (large J) approximation. Then trivial algebra gives,

$$\frac{1}{J+2}\delta_\psi^{q_2}\delta_\phi^{q_1}O^{J+2} = q_2\frac{1}{(J+2)^{3/2}(N/2)^{J+2}}\left(\sum_{l=0}^{J}(q_1q_2)^l\right)\sum_{p=0}^{J}q_2^p\,\mathrm{Tr}\left(\phi Z^p\psi Z^{J-p}\right).$$

The first sum vanishes unless $q_1q_2 = 1$. *Thus we reach at momentum conservation on the world-sheet*, $m = -n$. Also, one can simply omit the phase factor q_2 in front since the corresponding state is defined only up to a phase. Therefore one gets the BMN operator with two scalar impurities,

$$O^n_{\phi\psi} \equiv \frac{1}{J+2}\delta_\psi^{q_2}\delta_\phi^{q_1}O^{J+2} = \frac{1}{\sqrt{J(N/2)^{J+2}}}\sum_{p=0}^{J}e^{\frac{2\pi inp}{J}}\,\mathrm{Tr}\left(\phi Z^p\psi Z^{J-p}\right) \tag{2.6}$$

where we omitted $1/J$ corrections in large J approximation. Without the "dilute gas" approximation (for arbitrary J) one would get,

$$O^n_{\phi\psi} = \frac{1}{\sqrt{(J+2)(N/2)^{J+2}}}\left\{\sum_{p=0}^{J}e^{\frac{2\pi inp}{J+2}}\,\mathrm{Tr}(\phi Z^p\psi Z^{J-p}) + e^{-\frac{2\pi in}{J+2}}\,\mathrm{Tr}((\delta_\psi\delta_\phi Z)\,Z^{J+1})\right\} \tag{2.7}$$

instead of (2.6). In what follows, we will refer to the specific scalar impurity operator, (2.6) as the "scalar BMN operator".

Generalization to N string states in any transverse direction is obvious: Take corresponding N q_i-variations (which might be SO(6) transformation, translation or SUSY variation) of O^{J+N} where q_i are fixed as $q_i = e^{\frac{2\pi in_i}{J+N}}$, and momentum conservation $\sum nN_n = 0$ will be automatic.

2.2 Vector operator and conformal invariance

In this paper we will mainly be concerned with the vector operator which involves two impurity fields and constructed in analogy with the BMN operator (2.6) but with ψ impurity replaced with $D_\mu Z = \partial_\mu Z + ig[A_\mu, Z]$. It should be defined such that it reduces to a *descendant of the chiral primary operator*, $\partial_\mu\,\mathrm{Tr}[Z^{J+2}]$, when the phases are set to zero and it should be a *conformal primary* when the phases are present. Below, we will show that our general prescription for constructing BMN operators will do the job.

Before that, let us recall that two-point function of *conformal primary* vector operators $O_\mu(x)$ and $O_\nu(y)$ should have a specific transformation law under conformal transformations — particularly under inversion $x_\mu \to \frac{x_\mu}{x^2}$. The only possible tensorial dependence on x and y can be through the determinant of inversion,

$$J_{\mu\nu}(x-y) = \delta_{\mu\nu} - 2\frac{(x-y)_\mu(x-y)_\nu}{(x-y)^2}. \tag{2.8}$$

Therefore the two-point function is restricted to the form,

$$\langle O_\mu(x)O_\nu(y)\rangle \sim \frac{J_{\mu\nu}(x,y)}{(x-y)^{2\Delta}}$$

where Δ is the scale dimension. On the other hand, translation descendants of scalar conformal primaries $O_\mu(x) = \partial_\mu O(x)$, will have the following correlator,

$$\begin{aligned}\langle O_\mu(x)O_\nu(y)\rangle &\sim \partial_\mu\partial_\nu \frac{1}{(x-y)^{2(\Delta-1)}} \\ &= \frac{2(\Delta-1)}{(x-y)^{2\Delta}}\left(\delta_{\mu\nu} - 2\Delta\frac{(x-y)_\mu(x-y)_\nu}{(x-y)^2}\right).\end{aligned}$$

We would like to see whether the vector operator constructed with our prescription obeys these restrictions. We will not assume large J until we discuss correlators at the torus level and our construction will hold for *any* J. Therefore our prescription gives an analog of (2.7),

$$\begin{aligned}O^n_\mu \equiv \frac{1}{J+2}D^{q_2}_\mu \delta^{q_1}_\phi O^{J+2} &= \frac{1}{\sqrt{(J+2)(N/2)^{J+2}}} \times \\ &\times \left\{\sum_{l=0}^{J} q_2^l \operatorname{Tr}\left(\phi Z^l D_\mu Z Z^{J-l}\right) + q_2^{J+1}\operatorname{Tr}\left(D_\mu\phi Z^{J+1}\right)\right\}\end{aligned}$$

where D^q_μ is the gauge covariant "q-derivative" obeying the quantized derivation rule, (2.5). For $q=1$, q-derivation coincides with ordinary derivation. We will use the following two forms of vector operator interchangeably,

$$\begin{aligned}O^n_\mu &= \frac{1}{\sqrt{(J+2)(N/2)^{J+2}}}D^q_\mu \operatorname{Tr}\left(\phi Z^{J+1}\right) &(2.9)\\ &= \frac{1}{\sqrt{(J+2)(N/2)^{J+2}}}\left\{\sum_{l=0}^{J} q^l \operatorname{Tr}\left(\phi Z^l D_\mu Z Z^{J-l}\right) + q^{J+1}\operatorname{Tr}\left(D_\mu\phi Z^{J+1}\right)\right\} &(2.10)\end{aligned}$$

where,

$$q = e^{2\pi i n/(J+2)}\,. \tag{2.11}$$

Note that in (2.9) the position of ϕ would matter generally since the trace looses its cyclicity property under q-derivation. However one can easily check that only for the particular value (2.11), the cyclicity is regained: under an arbitrary shift, say by m units, in the position of ϕ, O^n_μ changes only by an overall phase q^m which is irrelevant to physics.

Having fixed the definition, it is now a straightforward exercise to compute the planar, tree level contribution to the two-point function $\langle O^n_\mu(x)\bar{O}^m_\nu(y)\rangle$ directly from (2.10) (or the one with ϕ shifted arbitrarily). Note that one can drop the commutator term in D_μ since it gives a $g^2_{\rm YM}$ correction to tree level. Denoting the scalar propagator by $G(x,y) = \frac{1}{4\pi^2(x-y)^2}$, the result is

$$\langle O^n_\mu(x)\bar{O}^m_\nu(y)\rangle = 2\delta_{nm}\frac{J_{\mu\nu}(x-y)}{(x-y)^2}G(x,y)^{J+2} \tag{2.12}$$

for arbitrary nonzero m, n. The appearance of inversion determinant $J_{\mu\nu}$, (2.8) clearly shows that vector operator is a conformal primary for arbitrary J, not necessarily large (at the tree level). In fact, a slight change in the definition of q (for example $q^J = 1$) would generate terms like $\mathcal{O}(1/J)\delta_{\mu\nu}$ which spoils conformal covariance for small J. Conformal covariance will be a helpful guide in the following calculations, therefore we shall stick to the definition, (2.11).

We also note that momentum on the world sheet is conserved at the planar level, but we will see an explicit violation at the torus-level just like in the case of scalar BMN operators. At this point we want to point out an alternative way to obtain above result directly by using (2.9) with $r = e^{2\pi i m/(J+2)}$:

$$\begin{aligned}\langle O^n_\mu(x)\bar{O}^m_\nu(y)\rangle &= \frac{1}{(N/2)^{J+2}(J+2)}\partial^q_\mu\partial^{\bar{r}}_\nu\langle \mathrm{Tr}(Z^{J+1}\phi)\,\mathrm{Tr}(\bar{Z}^{J+1}\bar{\phi})\rangle \\ &= \frac{1}{J+2}\partial^q_\mu\partial^{\bar{r}}_\nu\left(\frac{1}{(x-y)^{2(J+2)}}\right)\frac{1}{(4\pi^2)^{J+2}} \\ &= 2\delta_{nm}\frac{J_{\mu\nu}(x-y)}{(x-y)^2}G(x,y)^{J+2}\end{aligned}$$

using the planar tree level two-point function of chiral primaries and the definition of q-derivation, (2.5). This curious alternative way is a consequence of the fact that q-derivation and contraction operations commute with each other. We will use this fact to greatly simplify the calculations in the following sections. We also note that when the phases are absent above result trivially reduces to two-point function of translation descendants since q-derivation reduces to ordinary derivation for $q = 1$.

3. Free two-point function at torus level

Torus contribution to free two-point function of scalar BMN operators was calculated in [9]. Analogous calculation for vector operators is achieved simply by replacing one of the scalar impurities, say ψ field with the vector impurity $D_\mu Z$. Since our aim in this section is to obtain the free contribution, we can drop the commutator term in the covariant derivative which is $\mathcal{O}(g)$ and take the impurity as $\partial_\mu Z$. Consider the generic torus diagram in figure 1 where we show the ϕ-line together with $\partial_\mu Z$ impurity of the upper operator, O^J_μ, inserted at an arbitrary position and denoted by an arrow on a Z-line. This arbitrary position is to be supplied with the phase q^l and summed from $l = 0$ to $l = J+1$. To obtain $\langle O^J_\mu O^J_\nu\rangle_{\mathrm{torus}}$ one simply takes $\bar{r}$-derivative of this diagram. One should consider the following two cases separately.

First consider the case when $\bar{r}$-derivative hits the same Z-line as with ∂_μ. Then the phase summation will be identical to the phase sum for scalar BMN operators, which was outlined in section 3.3 of [9] with a $\mathcal{O}(1/J)$ modification coming from the fact that double derivative line can also coincide with the ϕ-line for vector operators. Here we will summarize the calculation of [9] for completeness. For simplicity let us first consider operators of same momentum, i.e. $q = r$. The double derivative line may be in any of the five groups containing $J_1 + \cdots + J_5 = J+2$ lines. If it is in the first or the last

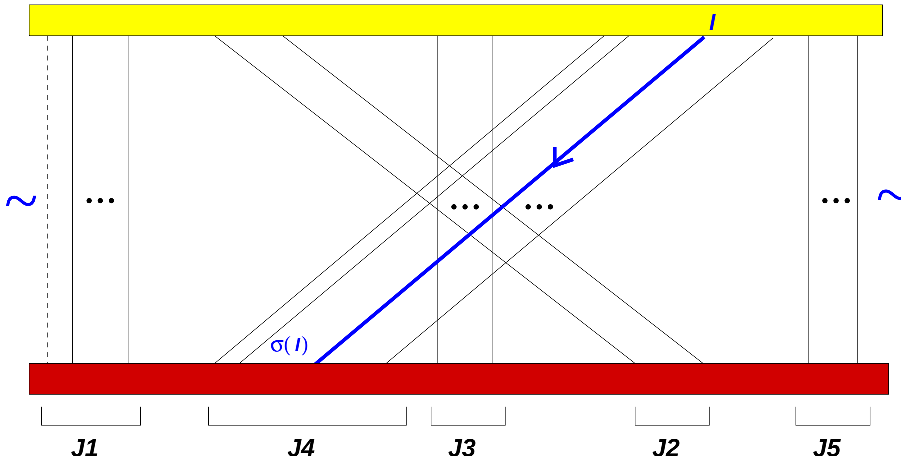

Figure 1: A typical torus digram. Dashed line represents ϕ and arrow on a solid line is $\partial_\mu Z$. The derivative ∂_ν can be placed on any line.

group of $J_1 + J_5$ possibilities, then there is no net phase associated with the diagram. If it is in any of the other three groups there will be a non-trivial phase, e.g. $q^{J_2+J_3}$ for the case shown in figure 1. Combining all possibilities the associated phase becomes $J_1 + J_2 q^{J_3+J_4} + J_3 q^{J_4-J_2} + J_4 q^{-J_2-J_3} + J_5$. One should sum this phase over all possible ways of dividing $J+2$ lines in five groups,

$$\begin{aligned}
&\frac{1}{(J+2)^5} \sum_{J_1+\ldots+J_5=J+2} \left(J_1 + J_2 q^{J_3+J_4} + J_3 q^{-J_4+J_2} + J_4 q^{J_2+J_3} + J_5\right) \\
&\to_{N\to\infty} \int_0^1 \mathrm{d}j_1 \cdots \mathrm{d}j_5 \, \delta(j_1 + \ldots + j_5 - 1)\times \\
&\times \left(j_1 + j_2 e^{-2\pi i n(j_3+j_4)} + j_3 e^{2\pi i n(-j_4+j_2)} + j_4 e^{2\pi i n(j_2+j_3)} + j_5\right) = \\
&\qquad = \begin{cases} \dfrac{1}{24}, & n = 0\,, \\ \dfrac{1}{60} - \dfrac{1}{6(2\pi n)^2} + \dfrac{7}{(2\pi n)^4} & n \neq 0\,. \end{cases}
\end{aligned} \tag{3.1}$$

In taking the limit $N \to \infty$ the fractions $j_i = J_i/(J+2)$ go over to continuous variables. Apart from this phase factor there is the obvious space-time dependence

$$\frac{1}{(x-y)^{2(J+1)}} \times f_{\mu\nu}$$

where $f_{\mu\nu} \equiv \partial_\mu \partial_\nu \frac{1}{(x-y)^2}$.

Now consider the second case when $\bar{q}$-derivative hits on a different Z-line than ∂_μ. For a fixed position of ∂_μ, say l, $\bar{r}$-derivative generates the phase sum

$$\sum_{l'=0;\, l'\neq\sigma(l)}^{J+1} \bar{q}^{l'} = -\bar{q}^{\sigma(l)}\,,$$

where we defined $\sigma(l)$ as the position at which $\partial_\mu Z$ connects the bottom operator, e.g. $\sigma(l) = l - (J_3 + J_4)$ for the case shown in figure 1, and used the definition $\bar{q}^{J+2} = 1$ to evaluate the sum over l'. Including the summation over l the total associated phase factor becomes,

$$-\sum_{l=0}^{J+1} q^l \bar{q}^{\sigma(l)} \tag{3.2}$$

which is obviously the same as (3.1) up to a minus sign. The associated space-time dependence is different however,

$$\frac{1}{(x-y)^{2J}} \times f_\mu f_\nu$$

where $f_\mu \equiv \partial_\mu \frac{1}{(x-y)^2}$. It is now easy to see that the torus phase factor of the vector and BMN operators will exactly be the same also for generic momenta m, n, not necessarily equal. This general phase factor was computed in [9] and we merely quote the final result,

$$A_{m,n} = \begin{cases} \frac{1}{24}, & m = n = 0\,; \\ 0, & m = 0, n \neq 0 \text{ or, } \; n = 0, m \neq 0\,; \\ \frac{1}{60} - \frac{1}{6u^2} + \frac{7}{u^4}, & m = n \neq 0\,; \\ \frac{1}{4u^2}\left(\frac{1}{3} + \frac{35}{2u^2}\right), & m = -n \neq 0\,; \\ \frac{1}{(u-v)^2}\left(\frac{1}{3} + \frac{4}{v^2} + \frac{4}{u^2} - \frac{6}{uv} - \frac{2}{(u-v)^2}\right), & \text{all other cases} \end{cases} \tag{3.3}$$

where $u = 2\pi m, u = 2\pi n$. Note also that space-time dependences of two separate cases that were considered above nicely combine into the conformal factor, (2.8), as

$$\frac{f_{\mu\nu}}{(x-y)^2} - f_\mu f_\nu = 2\frac{J_{\mu\nu}(x,y)}{(x-y)^6}\,. \tag{3.4}$$

Combining above results, free two-point function of the vector operators including genus one corrections can now be summarized as

$$\langle \bar{O}^m_\nu(y) O^n_\mu(x) \rangle_{\text{free torus}} = (\delta_{nm} + g_2^2 A_{nm}) \frac{2J_{\mu\nu}(x,y)}{(x-y)^2} G(x,y)^{J+2}\,. \tag{3.5}$$

This result clearly shows the mixing of O^n_μ operators at the torus level since the correlator is non-zero for $n \neq m$ (unless either n or m is zero). This operator mixing is described by the $\mathcal{O}(g_2^2)$ matrix $g_2^2 A_{nm}$. This particular momenta mixing issue of the BMN operators was first addressed in [7]. The eigenoperators corresponding to the true string eigenstates can be obtained diagonalizing the light cone hamiltonian at g_2^2 order. We will not need this diagonalization explicitly for our purposes. There is another type of mixing of the BMN operators at the torus level: single trace operators mix with the multitrace operators at $\mathcal{O}(g_2)$ [17]. Roughly, this corresponds to the fact that single string states are no longer the true eigenstates of the light-cone hamiltonian when one considers string interactions but mixing with multi-string states should be taken into account. As in [9] we shall ignore these mixing issues in this paper.

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4. Planar interactions of the two-point function

One of the main results of this manuscript is that vector operators possess the same anomalous dimension with the BMN operators. In this section we prove this result at the planar level and develop the techniques necessary to handle the interactions of vector operators which will also be used in the next section when we consider $\mathcal{O}(\lambda')$ interactions at genus one. These techniques can easily be used for $\mathcal{O}(\lambda')$ interactions at higher genera as well. However higher loop corrections would require non-trivial modifications.

Interactions of the vector operators are far more complicated than BMN operators because there are three new type of interactions that has to be taken into account. Recall $\mathcal{N} = 4$ SYM lagrangian (with euclidean signature) written in $\mathcal{N} = 1$ component notation [18],

$$\begin{aligned}\mathcal{L} = &\frac{1}{4}F_{\mu\nu}{}^2 + \frac{1}{2}\bar{\lambda}\not{D}\lambda + \overline{D_\mu Z^i} D_\mu Z^i + \frac{1}{2}\bar{\theta}^i \not{D}\theta^i + \\ &+ i\sqrt{2}gf^{abc}\left(\bar{\lambda}_a \bar{Z}^i_b L\theta^i_c - \bar{\theta}^i_a R Z^i_b \lambda_c\right) - \frac{g}{\sqrt{2}}f^{abc}\left(\epsilon_{ijk}\bar{\theta}^i_a L Z^j_b \theta^k_c + \epsilon_{ijk}\bar{\theta}^i_a R \bar{Z}^j_b \theta^k_c\right) - \\ &- \frac{1}{2}g^2\left(f^{abc}\bar{Z}^i_b Z^i_c\right)^2 + \frac{g^2_{\rm YM}}{2} f^{abc} f^{ade} \epsilon_{ijk}\epsilon_{ilm} Z^j_b Z^k_c \bar{Z}^l_d \bar{Z}^m_e \end{aligned} \tag{4.1}$$

where $D_\mu Z = \partial_\mu + ig[A_\mu, Z]$ and L, R are the chirality operators. For convenience we use complex combinations of the six scalar and fermionic fields in adjoint representation,

$$Z^1 = Z = \frac{X^5 + iX^6}{\sqrt{2}}, \qquad Z^2 = \phi = \frac{X^1 + iX^2}{\sqrt{2}}, \qquad Z^3 = \psi = \frac{X^3 + iX^4}{\sqrt{2}}$$

with analogous definitions for fermions, θ^i.

Recall the result of [9], (also see [18]) that, the only interactions involved in correlators of BMN operators were coming from F-terms since D-term and self-energy contributions exactly cancels each other out. That was due to a non-renormalization theorem for two-point functions of chiral primary operators and unfortunately, this simplification will no longer hold for the correlators involving vector operators because of the covariant derivatives. It will be convenient to group interactions in three main classes because the calculation techniques that we use will differ for each separate class:

1. D-term and self energies

2. Interactions of external gluons in O^n_μ

3. F-terms

In the following subsection we will show that interactions in the first class can be rewritten as a correlator of the non-conserved current, $\langle \mathrm{Tr}(J_\mu \bar{J}_\nu)\rangle$ where

$$J_\mu = Z \overset{\leftrightarrow}{\partial}_\mu Z\,. \tag{4.2}$$

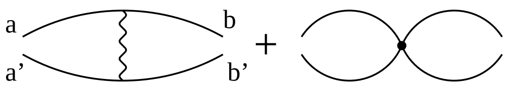
This will be a consequence of the non-renormalization theorem mentioned above. Second class of interactions which are coming from the commutator term in the covariant derivative will then promote the ordinary derivative in J_μ to a covariant derivative, hence total contribution of the interactions in first and second class will be represented as the correlator of a gauge-covariant (but non-conserved) current $U_\mu = Z \overleftrightarrow{D}_\mu Z$. Computation of this correlator by differential renormalization method [19] is given in appendix A. Last class of interactions originating from F-term in (4.1) are easiest to compute. This quartic vertex is only possible between a scalar impurity, ϕ and an adjacent Z-field (at planar level), and its contribution to the anomalous dimension of BMN operators was already computed in [9]. We can confidently conclude that F-term contribution to vector anomalous dimension is half the BMN anomalous dimension because the BMN operator involves two scalar impurities which contribute equally whereas the vector operator involves only one scalar impurity. However, a rigorous calculation for vector operators is provided in appendix B for completeness.

4.1 Non-renormalization of chiral primary correlator

Let us begin with recalling the non-renormalization theorem of the chiral primary correlator,

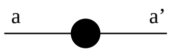
$$\langle \mathrm{Tr}(\phi Z^{J+1})\, \mathrm{Tr}(\bar{\phi}\bar{Z}^{J+1})\rangle \,. \tag{4.3}$$

For our purposes it will suffice to confine ourselves to planar graphs. D-term part of (4.1) includes the quartic interaction $\mathrm{Tr}([Z,\overline{Z}]^2)$ (or $\mathrm{Tr}([\phi,\overline{\phi}][Z,\overline{Z}])$) and the gluon exchange between two adjacent Z-lines,



$$= (f^{pab} f^{pa'b'} + f^{pab'} f^{pa'b}) B(x,y) G(x,y)^2 \tag{4.4}$$

where a, a', b, b' indicate adjoint color indices, $G(x,y) = 1/(4\pi^2(x-y)^2)$ is the free scalar propagator and $B(x,y)$ is a function which arise from the integration over vertex positions and contains information about the anomalous dimension. Self-energy corrections to Z and ϕ propagators arise from a gluon exchange, chiral-chiral and chiral-gaugino fermion loops. We represent this total self-energy contribution as,



$$= \delta^{aa'} N A(x,y) G(x,y) \tag{4.5}$$

where $A(x,y)$ again contains $\mathcal{O}(g^2_{\mathrm{YM}})$ contribution to anomalous dimension.

Planar contributions to (4.3) are obtained by inserting (4.4) in between all adjacent Z-Z and ϕ-Z pairs and summing over self energies on all J lines including ϕ. Since every term in (4.1) is flavor blind except than the F-term, eqs. (4.4) and (4.5) also hold for ϕ. Therefore, from now on we do not distinguish interactions of ϕ and Z fields and in all of the following figures a solid line represents either ϕ or Z (unless ϕ is explicitly shown by a dashed line).

A convenient way to represent sum of all these interactions is to define a total vertex as shown in figure 2 and sum over $J+2$ possible insertions of this vertex in between all adjacent lines.

Note that in figure 2 the self-energy contributions on each line are taken as half the original value, $A(x,y)/2$, to compensate the double-counting of self energies by this method. One of the $J+2$ possible contributions is shown in figure 3. Using the trace identities given

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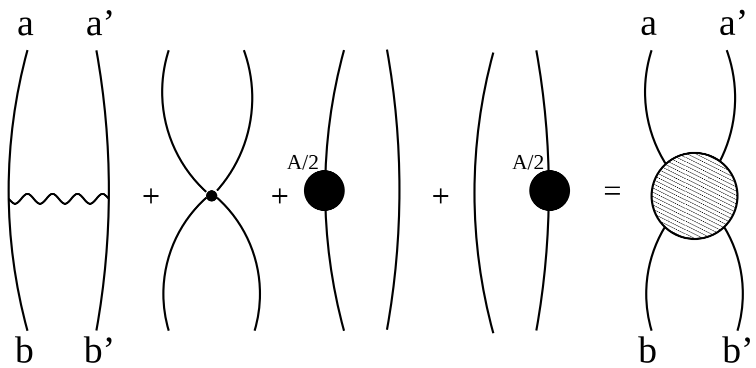

Figure 2: Combination of $g^2_{\rm YM}$ corrections under a total vertex.

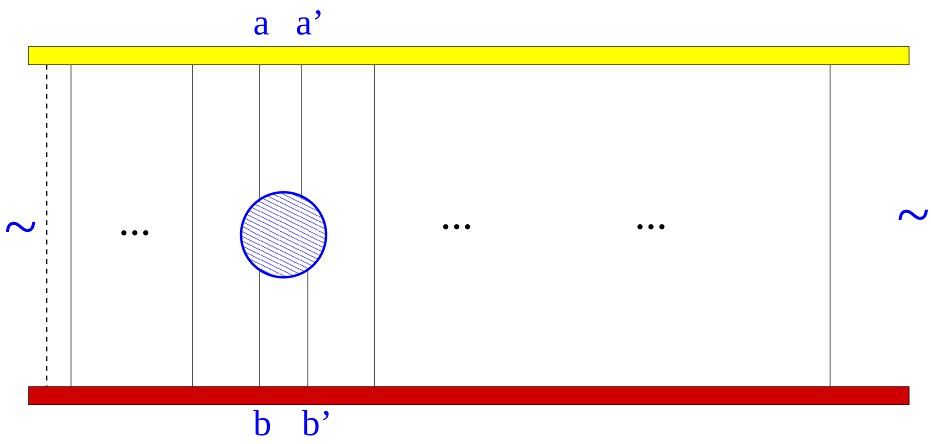

Figure 3: Planar interactions of chiral primaries can be obtained by placing the total vertex between all adjacent pairs. To find the vector correlator one simply dresses this figure by ∂^q_μ and $\partial^{\bar r}_\nu$.

in appendix B. it is straightforward to compute the amplitude represented by figure 3. One obtains,

$$\begin{aligned}
\text{Figure 3} \;=\; & G(x,y)^J \left\{ \frac{1}{2}\left(\frac{N}{2}\right)^{J-3} \mathrm{Tr}\left(T^a T^{a'} T^b T^{b'}\right)\left(f^{pab} f^{pa'b'} + f^{pab'} f^{pa'b}\right) \times \right. \\
& \left. \times\, B(x,y) + \left(\frac{N}{2}\right)^{J+1} A(x,y) \right\} \\
=\; & G(x,y)^J \left(\frac{N}{2}\right)^{J+1} \left\{ B(x,y)(1+\frac{2}{N^2}) + A(x,y) \right\} \\
\to\; & G(x,y)^J \left(\frac{N}{2}\right)^{J+1} \{B(x,y) + A(x,y)\}\,.
\end{aligned}$$

$\mathcal{O}(1/N^2)$ term in second line is coming from the second permutation in (4.4) and is at torus order hence negligible in the BMN limit taken in the last line. Clearly, insertions in

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all other spots give equal contributions and the final answer becomes,

$$\left\langle \mathrm{Tr}(\phi Z^{J-1})\,\mathrm{Tr}\left(\overline{\phi Z}^{J-1}\right)\right\rangle = G(x,y)^J (J+2)\left(\frac{N}{2}\right)^{J+1}\{B(x,y)+A(x,y)\}\,. \tag{4.6}$$

Then, the non-renormalization theorem of this correlator [20, 18] tells that,

$$4.6 \propto \mathrm{Tr}\left(\quad \right) \;\propto\; (B \;+\; A) = 0\,. \tag{4.7}$$

where the symbol in the middle is the total vertex in figure 2. This identity greatly simplifies the following calculations.

4.2 D-term and self-energies

Now, consider the D-term and self-energy contributions to planar two-point function of vector operators,

$$\left\langle O^n_\mu(x)\overline{O}^m_\nu(y)\right\rangle_1 = \partial^q_\mu \partial^{\bar r}_\nu \left\langle \mathrm{Tr}(\phi Z^{J+1})\,\mathrm{Tr}\left(\overline{\phi Z}^{J+1}\right)\right\rangle \tag{4.8}$$

where we used the fact that q-derivation and commutation operations commute with each other to take q-derivatives out of the correlator. Once again, we note that minimal coupling in the covariant derivative can be dropped as its contribution will be of order $\mathcal{O}(g^3)$. Now it is clear that calculation is reduced to taking $\partial^q_\mu\partial^{\bar r}_\nu$ of figure 3 and summing over all possible locations of the total vertex in figure 3. In taking $\partial^q_\mu\partial^{\bar r}_\nu$ of figure 3, one encounters three possibilities.

If both of the derivatives hit lines other than the four legs coming into the total vertex, than graph is proportional to $(B+A)\partial_\mu G(x,y)\partial_\nu G(x,y)$ hence vanishes by (4.7). Second possibility is when one of the q-derivatives hit the vertex and other outside. Supposing ∂^q_μ hits the vertex, trivial algebraic manipulations show that the graph will be proportional to,

$$\begin{aligned}\partial_\nu G(x,y)\partial^q_\mu\,\mathrm{Tr}\left\{\quad\right\} &= \frac{1}{2}\partial_\nu G(x,y)\Big[(1-q)\,\mathrm{Tr}\left\{\quad - \quad\right\} + \\ &\qquad + (1+q)\partial_\mu\,\mathrm{Tr}\left\{\quad\right\}\Big] \\ &= \frac{1}{2}\partial_\nu G(x,y)\Big[(1{-}q)\langle \mathrm{Tr}(J_\mu \bar Z\bar Z)\rangle {+} (1+q)\partial_\mu\left\{(B{+}A)G(x,y)^2\right\}\Big]\,.\end{aligned}$$

where J_μ was defined in (4.2). The second term in the last line again vanishes by (4.7) whereas the first term is the self energy and D-term corrected two point function of a vector operator J_μ with a scalar operator $\bar Z^2$. Now, it is immediate to see that by the antisymmetry of derivative in J_μ, both D-term quartic vertex and self ebergy corrections to $\langle J_\mu \bar Z^2\rangle$ vanishes. With a little more afford one can also see that the gluon exchange contribution is identically zero as well and the second possibility givevs no contribution.

Therefore we are only left with the third possibility where both ∂^q_μ and $\partial^{\bar r}_\nu$ are acting on the vertex in figure 3. With similar algebraic manipulations of this graph and the use of (4.7) one obtains,

$$\partial^q_\mu\partial^{\bar r}_\nu\,\mathrm{Tr}\left\{\quad\right\} = \frac{1}{4}(1-q)(1-\bar r)\langle \mathrm{Tr}(J_\mu(x)\overline{J}_\nu(y))\rangle. \tag{4.9}$$

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Recall that there is a phase factor depending on the position of the vertex in figure 3. If this position is l then this factor equals $(q\bar{r})^l$ and one should sum over the vertex position from $l = 0$ to $l = J + 1$ to obtain the total contribution. Using our definition of the vector phase, (2.11), this phase summation generates the multiplicative factor $(J+2)\delta_{mn}$. Furthermore, use of the trace identities of appendix B one squeezes the whole trace down to the trace of interacting part with a multiplicative factor of $\frac{1}{2}(N/2)^{J-1}$ (See appendix B for a similar application of the trace identities). The final answer can be written as,

$$\begin{aligned} \langle O^n_\mu(x)\overline{O}^m_\nu(y)\rangle_1 &= \partial^q_\mu \partial^{\bar{r}}_\nu \left\langle \mathrm{Tr}(\phi Z^{J+1})\,\mathrm{Tr}\left(\overline{\phi Z}^{J+1}\right)\right\rangle_1 \\ &= G(x,y)^J \frac{1}{2}\left(\frac{N}{2}\right)^{J-1}(J+2)\delta_{mn}\frac{1}{4}(1-q)(1-\bar{r})\left\langle \mathrm{Tr}\left(J_\mu(x)\overline{J}_\nu(y)\right)\right\rangle . \end{aligned} \tag{4.10}$$

Radiative corrections to this current correlator arise form three sources: D-term quartic vertex, gluon exchange and self energies. It is easy to see that D-term contribution vanishes identically by the antisymmetry of J_μ under exchange of two incoming Z particles. This is shown in appendix A. Self-energy contributions are straightforward to calculate with Differential Renormalization method [19] and calculations are explicitly shown in appendix A.

However, gluon exchange contribution to $\langle J\bar{J}\rangle$ is notoriously difficult to evaluate by direct methods. Fortunately, there is the following trick:[5] suppose that one computes the true flavor-current correlator of scalar QED, $\langle j\bar{j}\rangle$ with $j = \bar{Z}\overleftrightarrow{\partial} Z$ instead. Feynman rules treat these two correlators equivalently except than an overall minus sign (J and j differs only by replacement of a Z with a $\bar{Z}$ then color factors at external vertices give rise to a minus sign) and the appropriate color factors at the vertices. Hence one can obtain the anomalous dimension which arise from gluon exchange graph by considering the vacuum polarization graphs of scalar QED at two-loop order as we exlain below. There are four Feynman diagrams that are shown in figure 4. Note that, the anomalous dimension arises from the sub-divergent pieces of the gluon exchange graph (when the internal vertices come close to x or y.) Now, the Ward identity of scalar QED requires that the sub-divergent logaritmic pieces of graph I, II and III cancels each other out (IV do not contribute to anomalous dimension.) This fact allows us to compute gluon exchange in terms of I and II which are easy enough to evaluate directly as shown in appendix A. When the smoke clears one obtains the total anomalous dimension arising from D-term and self-energy part of the lagrangian as,

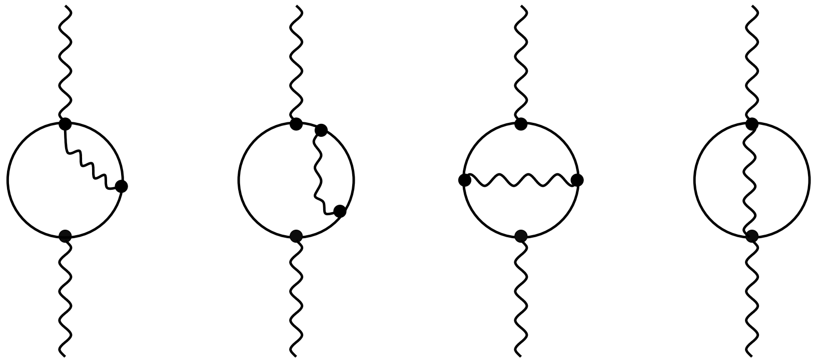
$$\langle O^n_\mu(x)\overline{O}^m_\nu(y)\rangle_1 = -\frac{5}{8}\lambda' n^2 \delta_{mn}\log\left((x-y)^2\Lambda^2\right)\frac{J_{\mu\nu}(x,y)}{(x-y)^2}G(x,y)^{J+2} \tag{4.11}$$

with the correct normalization of the vector operators.

4.3 External gluons

There are two topologically different classes of planar diagrams which involve external gluons at $\mathcal{O}(g^2_{\mathrm{YM}})$ order. First class, shown in figure 5, which arise from contracting external gluons of O^n_μ and $\overline{O}^m_\nu$ do not involve any internal vertex to be integrated over,

[5] We are grateful to Dan Freedman for pointing out this idea.



I II III IV

Figure 4: Two-loop diagrams of vacuum polarization in scalar QCD. Treatment of other four diagrams obtained by replacing the scalar lines with gluons can be separately and do not affect our argument.

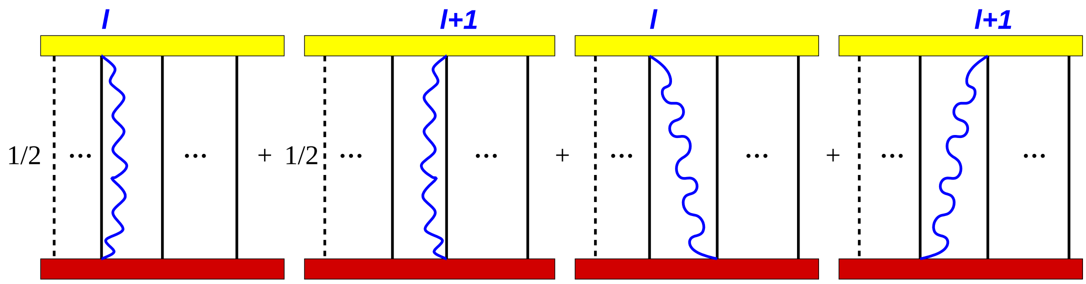


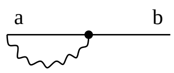
Figure 5: First class of $\mathcal{O}(g_{\rm YM}^2)$ diagrams involving external gluons. These do not yield anomalous dimension as there are no internal vertices.

hence do not give rise to log terms. By considering all possible Wick contractions and employing the trace identities of appendix B, sum over all of these diagrams yield (in Feynman gauge),

$$\langle O^n_\mu(x)\overline{O}^m_\nu(y)\rangle_2 \to g_{\rm YM}^2(1-q)(1-\bar{r})\delta_{\mu\nu}\frac{\delta_{mn}}{(x-y)^2}(J+2)\left(\frac{N}{2}\right)^{J+3} G(x,y)^{J+2}\,. \qquad (4.12)$$

Therefore this class does not contribute to the anomalous dimension.

Second class of diagrams which involve one external gluon and one internal cubic vertex are depicted in figure 6. Diagrams where external gluon belongs to $\overline{O}^m_\nu$ will give identical contributions to those in figure 6, hence need not be considered separately. Minimal coupling in the covariant derivative in $\overline{O}^m_\nu$ can again be dropped since we are interested in $\mathcal{O}(g_{\rm YM}^2)$. Let us first consider graph I in figure 6. Total contribution to correlator is obtained by taking $\partial^{\bar{r}}_\nu$ of this diagram and performing the phase sum over all possible positions $l \in \{0,\ldots,J+1\}$. Define,



$$= N\delta_{ab}C_\mu(x,y)\,. \qquad (4.13)$$

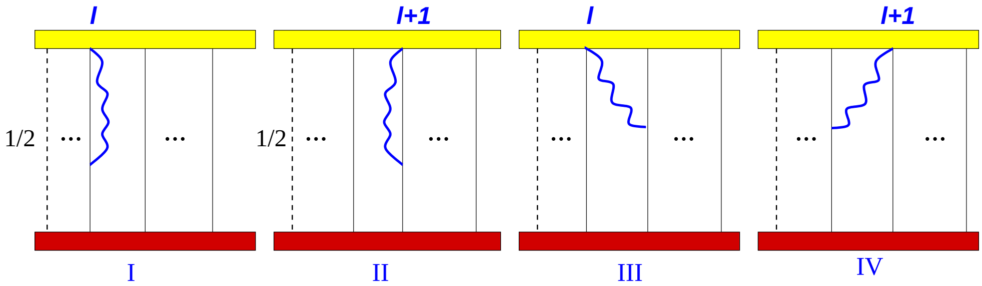


Figure 6: Second class of diagrams which involve one external gluon. Derivative of the $\bar{O}^J_\nu$ can be placed at any position. Integration over the internal vertex yields a contribution to anomalous dimension.

Using the identity, $\bar{r}^{J+2} = 1$ one easily obtains the $\partial^{\bar{r}}_\nu$ of Graph I with the result,

$$(q\bar{r})^l G(x,y)^J \frac{1}{4}\left(\frac{N}{2}\right)^J N^3 \left\{\partial_\nu C_\mu(x,y)\, G(x,y) - \partial_\nu G(x,y) C_\mu(x,y)\right\}$$

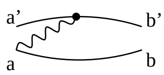
where we again used the trace identities of appendix B. Summation over l yields,

$$G(x,y)^J \frac{1}{4}\left(\frac{N}{2}\right)^J N^3 (J+2)\delta_{mn}\left\{G(x,y)\ \overleftrightarrow{\partial}_\nu\ C_\mu\right\}. \tag{4.14}$$

Graph II gives identical contribution except than a factor of $q\bar{r}$. In appendix A, we compute $C_\mu(x,y)$ and conclude that graphs I and II give the following contribution to the anomalous dimension (including the equal contribution from the reflected graph where external gluon belongs to $\overline{O}^m_\nu$),

$$\text{Graph I} + \text{II} \rightarrow -\frac{3}{8}(1+q\bar{r})\frac{g^2_{\rm YM}N}{4\pi^2}\delta_{mn}\frac{J_{\mu\nu}(x,y)}{(x-y)^2}\log\left((x-y)^2\Lambda^2\right)G(x,y)^{J+2}. \tag{4.15}$$

To handle graph III in figure 6, let us write,



$$= -(f^{pab}f^{pa'b'} + f^{pab'}f^{pa'b})C_\mu(x,y)G(x,y). \tag{4.16}$$

By the same token, $\partial^{\bar{r}}_\nu$ of graph III can be written as (at planar level),

$$\begin{aligned} &-\frac{1}{2}\left(\frac{N}{2}\right)^{J-1} G(x,y)^{J-1}\left(\frac{N^4}{8}\right)(q\bar{r})^l \times \\ &\times\left\{\partial^{\bar{r}}_\nu(G(x,y)C_\mu(x,y))G(x,y) - (1+\bar{r})\partial_\nu G(x,y)G(x,y)C_\mu(x,y)\right\} = \\ &\qquad = \frac{1}{2}\left(\frac{N}{2}\right)^{J-1} G(x,y)^J (q\bar{r})^l \bar{r}\left(\frac{N^4}{8}\right)\left\{G(x,y)\ \overleftrightarrow{\partial}_\nu\ C_\mu(x,y)\right\} \end{aligned} \tag{4.17}$$

where $N^4/8$ comes from the the color factor, $f^{pab}f^{pa'b'}\,{\rm Tr}(T^aT^{a'}T^{b'}T^b)$. Second color combination in (4.16) gives torus level contribution hence negligible at planar level. Summing over l and using the expression for C_μ which is evaluated in appendix A, one gets the following contribution,

$$\text{Graph III} \to -\frac{3}{8}\frac{g_{\rm YM}^2 N}{4\pi^2}\bar{r}\delta_{mn}\frac{J_{\mu\nu}(x,y)}{(x-y)^2}\log\left((x-y)^2\Lambda^2\right)G(x,y)^{J+2} \tag{4.18}$$

where we included the equal contribution coming from the horizontal reflection of graph IV. Graph IV and horizontal reflection of III gives (4.18) with $\bar{r}$ is replaced by q, giving all in all,

$$\begin{aligned}\langle O_\mu^J(m;x)\overline{O}_\nu^J(n;y)\rangle_2 &= -\frac{3}{8}\frac{g_{\rm YM}^2 N}{4\pi^2}(1-q-\bar{r}+q\bar{r})\delta_{mn}\log\left((x-y)^2\Lambda^2\right)\frac{J_{\mu\nu}(x,y)}{(x-y)^2}G(x,y)^{J+2} \\ &= -\frac{3}{8}\frac{g_{\rm YM}^2 N}{4\pi^2}(1-q)(1-\bar{r})\delta_{mn}\log\left((x-y)^2\Lambda^2\right)\frac{J_{\mu\nu}(x,y)}{(x-y)^2}G(x,y)^{J+2} \\ &= -\frac{3}{8}\lambda' n^2\delta_{mn}\log\left((x-y)^2\Lambda^2\right)\frac{J_{\mu\nu}(x,y)}{(x-y)^2}G(x,y)^{J+2} \end{aligned} \tag{4.19}$$

as the total contribution to anomalous dimension from external gluons, after normalizing according to (2.9).

As an aside let us make an important observation which will be used in section 4. In the previous section we concluded that D-term and self-energy contributions to the vector correlator can be organized in terms of the current two-point function $\langle J_\mu \bar{J}_\nu\rangle$ where $J_\mu = Z \overleftrightarrow{\partial}_\mu Z$. Curiously enough, the external gluons result, eq. (4.19) can exactly be reproduced (in order $\mathcal{O}(g_{\rm YM}^2)$) by promoting the ordinary derivative of J_μ in (4.10) to the covariant derivative. Therefore, *one can neatly represent the contributions of D-term, self-energy and external gluons to the anomalous dimension in terms of radiative corrections to the current correlator* $\langle \mathrm{Tr}(U_\mu(x)\overline{U}_\nu(y))\rangle$ where U_μ is the *gauge-covariant* but *non-conserved current*, $U_\mu = Z \overleftrightarrow{D}_\mu Z$. Had U_μ been conserved there would not be any radiative corrections to the correlator and the corresponding non-F term contributions to anomalous dimension would vanish identically. By using the equations of motion one can easily see that the "non-conservation" of U_μ is of $\mathcal{O}(g_{\rm YM})$ hence one expects first order corrections to $\langle U\bar{U}\rangle$ be $\mathcal{O}(\lambda')$. Indeed, combining (4.11) and (4.19) one obtains,

$$\langle O_\mu^n(x)\overline{O}_\nu^m(y)\rangle_{1+2} = G(x,y)^J\delta_{mn}\frac{1}{4}(1-q)(1-\bar{r})\langle \mathrm{Tr}(U_\mu(x)\overline{U}_\nu(y))\rangle \tag{4.20}$$

$$\to -\lambda' n^2\delta_{mn}\log\left((x-y)^2\Lambda^2\right)\frac{J_{\mu\nu}(x,y)}{(x-y)^2}G(x,y)^{J+2} \tag{4.21}$$

Notice that anomalous dimension is in units of the correct effective 't Hooft coupling associated with the BMN limit i.e. $\lambda' = g_{\rm YM}^2 N/J^2$, and tensorial form of the correlator indicates that conformal primary nature of O_μ^J operators is preserved by planar radiative corrections.

This result proves our previous claim that F-term contribution (which is calculated in appendix B) and the rest (D-term, self-energy and external gluons) are equal hence the total planar two-point function of vector operators with $\mathcal{O}(\lambda')$ radiative corrections can be written as,

$$\langle O_\mu^n(x)\overline{O}_\nu^m(y)\rangle = \left(1-\lambda' n^2\log\left((x-y)^2\Lambda^2\right)\right)\delta_{mn}\frac{2J_{\mu\nu}(x,y)}{(x-y)^2}G(x,y)^{J+2} \tag{4.22}$$

which shows that *the vector operators possess the same anomalous dimension as BMN operators, as required by the consistency of the BMN conjecture.* This concludes our first test on the BMN conjecture.

5. Anomalous dimension on the torus

In the previous section we noted that F-term contribution to planar anomalous dimension of vector operators is just half of the BMN case because vector operators involve one scalar impurity field compared to two impurities of BMN operators. Similarly, one can easily show the effect of F-term interactions on the torus which arise from the ϕ impurity produces half of the BMN torus dimension. Furthermore, we will show that D-term and external gluon contributions combine neatly into the form $\langle \mathrm{Tr}(U_\mu \bar{U}_\nu)\rangle$ as in the planar case, hence yield the same torus anomalous dimension as the F-terms. Therefore, the total torus anomalous dimension of BMN and vector operators are the same as well.

As in the previous section we will group the interactions into D-term, external gluon and F-terms but before that it is convenient to classify topologically different torus diagrams which will show up in each of these interaction classes. Notice that all of these interactions will result in two interaction loops which are in contact with each other at the interaction vertex, figure 7. Therefore, one can classify torus diagrams [9] which are leading order in J, i.e. $\mathcal{O}(J^3)$, according to whether

1. both of the loops are contractible,
2. only one is non-contractible,
3. both are non-contractible on the same cycle of torus,
4. both are non-contractible on different cycles of torus.

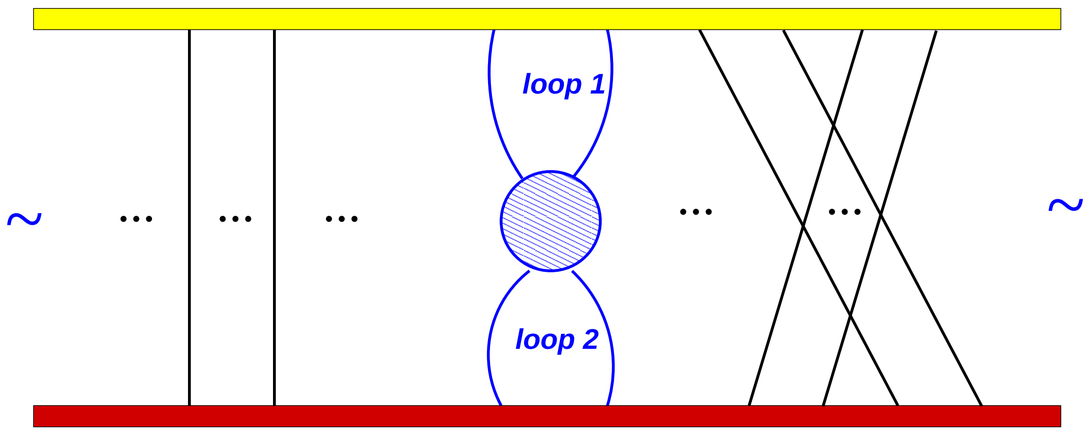


Figure 7: A generic $\mathcal{O}(g_{\mathrm{YM}}^2)$ interaction on a torus diagram. The internal vertex generates two interaction loops in space-time graphs.

We will call these groups as *contractible*, *semi-contractible*, *non-contractible* and *special* respectively. We called the last class *special* because it is possible only for D-term interactions as we demonstrate below. First three of these classes were discussed in [9] in detail where they were called as *nearest*, *semi-nearest* and *non-nearest* respectively. In what follows, we shall demonstrate that only the "non-contractible" class gives rise to a torus anomalous dimension.

5.1 Contractible diagrams

A generic contractible diagram is displayed on the cylinder and on the periodic square in figures 8 and 9 respectively. As in planar interactions, we combined D-term quartic vertex with gluon exchange and self energies into the total vertex, see figure 2.

Let us first consider the contractible contribution to the chiral primary correlator, $\langle O^J_\phi \bar{O}^J_\phi \rangle$ where O^J_ϕ is defined in (2.3). This of course will vanish by the non-renormalization theorem of section 4.1. Still, it will be helpful for illustrative purposes to discuss this case first because we will obtain $\langle O^n_\mu \overline{O}^m_\nu \rangle$ by taking q-derivatives of the chiral primary correlator. To obtain this contribution we would sum the diagrams similar to figure 8 with all possible contractible insertions of the this vertex. By the use of trace identities of appendix B, one gets,

$$\langle O^J_\phi(x) \bar{O}^J_\phi(y) \rangle \propto (\, B(x,y) + A(x,y)\,) G(x,y)^{J+2} N^{J+1} J^5 \,. \tag{5.1}$$

Total vertex gives the $B+A$ factor as in (4.6), power of N indicates that this is a torus level (to be compared with N^{J+3} dependence at planar level) and the dependence on J is coming from two observations: there are $\sim J^4$ free diagrams that can be drawn on a torus and the interaction vertex can be inserted at $\sim J$ different positions respecting the contractibility of the diagram. This, of course vanishes by the non-renormalization theorem, (4.7). One obtains correlator of vector operators simply by taking ∂^q_μ and $\partial^{\bar{r}}_\nu$ of figure 8. By the same reasoning as in our planar calculation, one sees that the only non-vanishing case occurs

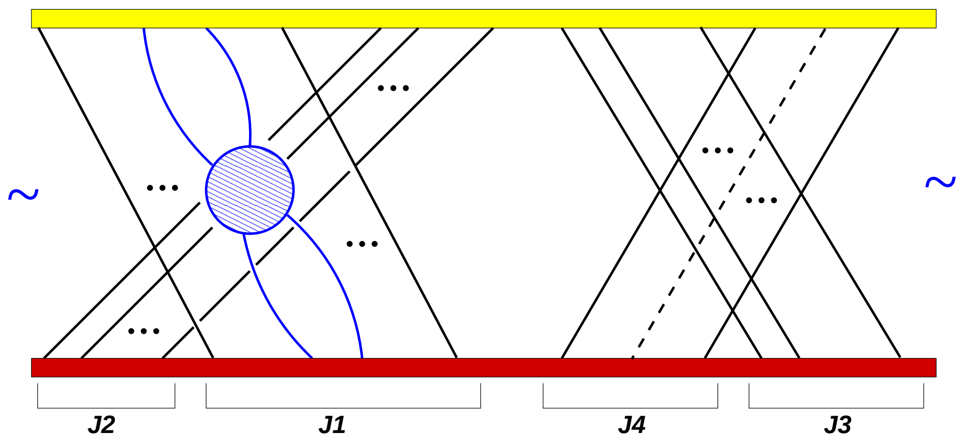


Figure 8: A generic contractible diagram. Total vertex includes, D-term quartic vertex, gluon exchange and self-energy corrections.

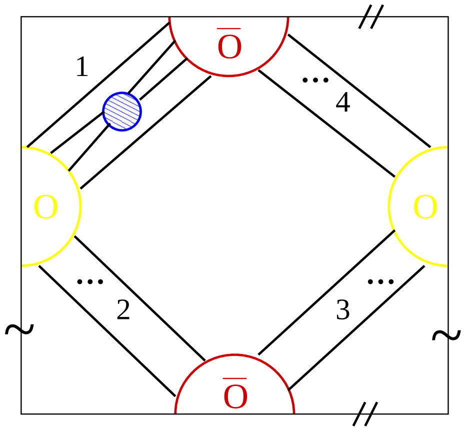


Figure 9: Same diagram as figure 8, but represented on a periodic square.

when both of the derivatives hit legs of the total vertex. In that case one arrives at the following expression,

$$\frac{1}{4}q^l\bar{r}^{\sigma(l)}(1-q)(1-\bar{r})\langle \mathrm{Tr}(J_\mu \bar{J}_\nu)\rangle G^J N^{J-3}\,.$$

Phase summation $\sum_{l=0}^{J+1} q^l\bar{r}^{\sigma(l)}$ is identical to (3.2) yielding the phase factor, $A_{n,m}$, in eq. (3.5). As we are interested in the anomalous dimension, we consider the case $n=m$ and get,

$$\langle O^n_\mu(x)\overline{O}^n_\nu(y)\rangle = \frac{A_{n,n}}{4}(1-q)(1-\bar{r})\langle \mathrm{Tr}(J_\mu \bar{J}_\nu)\rangle G^J N^{J-3}. \tag{5.2}$$

Apart from the phase factor associated with the topology of these diagrams the calculation is identical to the planar case. Therefore, together with the contribution from external gluons and F-term quartic vertex (which is only possible between ϕ and adjacent Z's) the final answer can be written as,

$$\langle O^n_\mu(x)\bar{O}^n_\nu(y)\rangle_{\text{contractible}} = -g_2^2\lambda' n^2 \log\left[(x-y)^2\Lambda^2\right] A_{nn}\frac{2J_{\mu\nu}(x,y)}{(x-y)^2}G(x,y)^{J+2}\,. \tag{5.3}$$

where we included the normalization associated with the torus correlator. We conclude that contractible diagrams *do not* contribute to torus anomalous dimension because their sole effect is to modify the normalization of the two point function by the factor of $A_{n,n}$.

5.2 Semi-contractible diagrams

An example of the second class of diagrams which might potentially contribute to torus anomalous dimension is shown on the cylinder and the periodic square in figures 10 and 11. However, we will now show that sum over all possible semi-contractible diagrams actually vanishes. Last figure explicitly shows that the interaction loop which is formed by two adjacent Z lines connected to O^n_μ is contratible whereas the other interaction loop formed by $\bar{Z}$ lines connected to O^m_ν is surrounding a cycle of the torus. A glance at either figures show that there are 8 possible positions that one can put in such a semi-contractible vertex

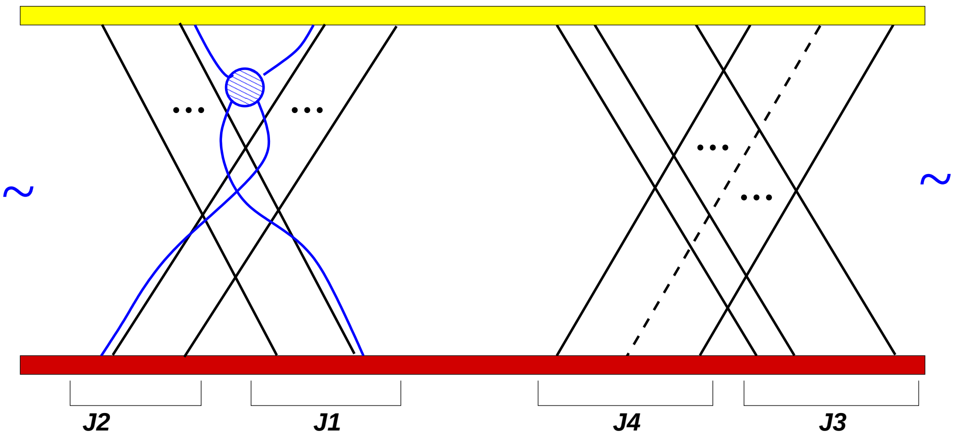


Figure 10: A semi-contractible diagram shown on the cylinder.

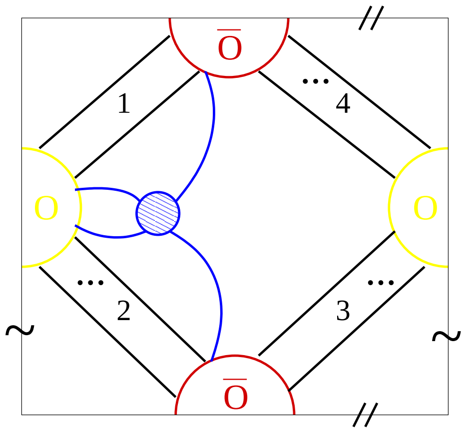


Figure 11: Same diagram as figure 10, but represented on a periodic square.

— as opposed to J possible insertions of contractible vertex — hence the multiplicity of this class of graphs is order J less than the contractible class, that is $\mathcal{O}(J^4)$. As we explain next, the phase summation provides a factor of $O(1/J)$ rendering semi-contractible class leading order in J i.e. $O(J^3)$.

To compute the contribution of figure 10 to the total anomalous dimension one can use the same trick as above. One first combines D-term, gluon exchange and self energies under the total vertex of figure 2. Then, insertion of derivatives in all possible ways with the phases shows that,

$$\langle O^n_\mu(x)\bar{O}^m_\nu(y)\rangle \propto (1-q)(1-\bar{r}^{J_1+J_2})\langle \mathrm{Tr}(J_\mu \bar{J}_\nu)\rangle$$

for the fixed position of ϕ as in figure 10 and fixed $J_1 \ldots J_4$. Contributions from the external gluons are shown in figure 12. Not surprisingly, they add up with total vertex to modify the above result as,

$$\langle O^n_\mu(x)\bar{O}^m_\nu(y)\rangle \propto (1-q)(1-\bar{r}^{J_1+J_2})\langle \mathrm{Tr}(U_\mu \bar{U}_\nu)\rangle\,.$$

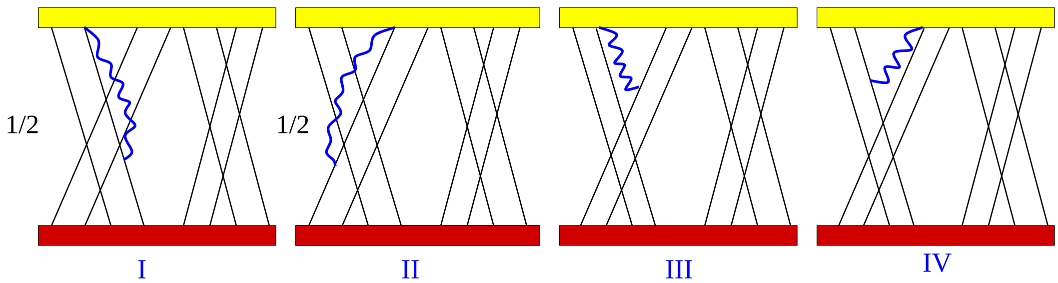


Figure 12: External gluon interactions with semi-contractible topology.

One similarly computes the contributions from 7 other semi-contractible graphs with the *same* $J_1 \ldots J_4$ and the *fixed* position of ϕ, and finds that phase factors conspire exactly to cancel out the total result. F-term contributions also give rise to same phase factors and cancel out in exactly the same way as described above.

5.3 Non-contractible diagrams

As advertised in the beginning of this section, we will now show that non-contractible diagrams, figures 13 and 14, yield a finite contribution to torus anomalous dimension. As one can observe in figure 14, to join the legs of the interaction vertex while both interaction loops surround the same cycle of the torus, it is necessary that one of the 4 possible Z-blocks be absent (block 3 is absent in figure 14). Therefore the multiplicity of this class is $1/J$ lower than the semi-contractible class, that is $\mathcal{O}(J^3)$. However phase summation will not change this order essentially because upper and lower legs of the interaction vertex are separated by a macroscopic number of Z lines. One concludes that non-contractible diagrams are also $\mathcal{O}(J^3)$ i.e. leading order.

Figure 15 shows non-contractible external gluon diagrams. Having gained experience with previous calculations one can immediately write down the contribution of the total

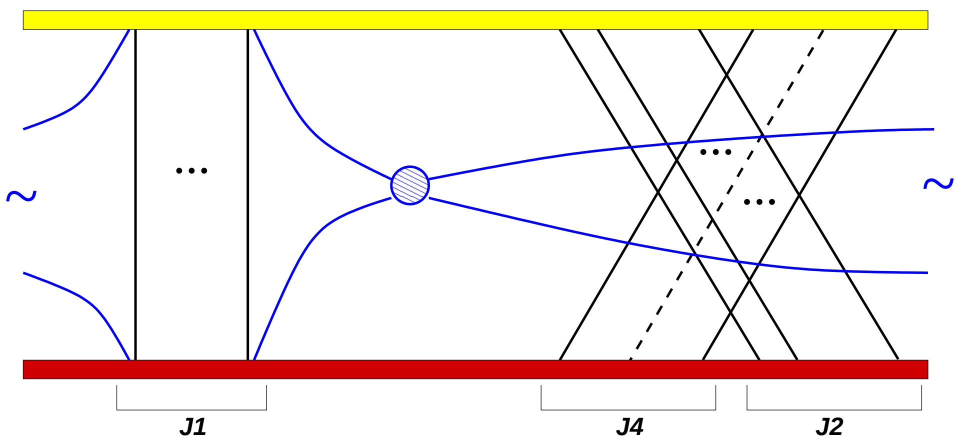


Figure 13: Non-contractible diagrams on the cylinder.

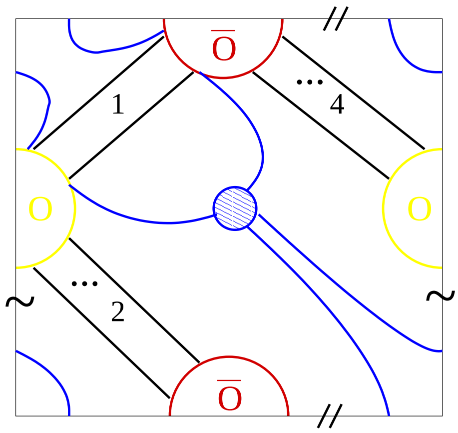


Figure 14: A Non-contractible diagram on the periodic square. Note that 3rd block of Z-lines is missing.

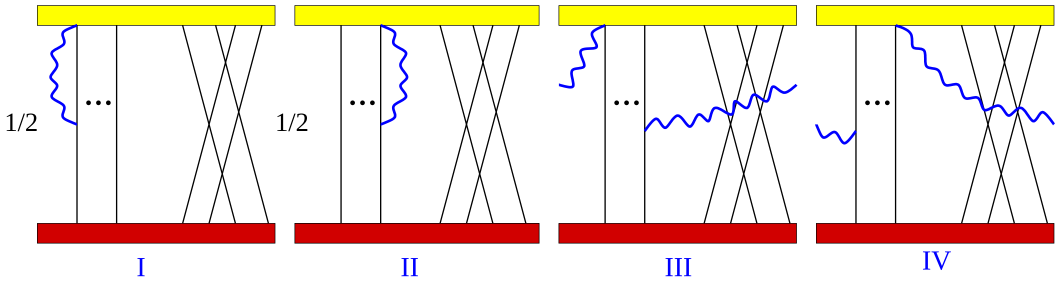


Figure 15: External gluon interactions with non-contractible topology.

D-vertex and external gluons (for the fixed position of ϕ shown in figures) as,

$$\frac{1}{4}(1-q^{J_1})(1-\bar{q}^{J_1})\bar{q}^{J_2}\langle \mathrm{Tr}(U_\mu(x)\bar{U}_\nu(y))\rangle G(x,y)^J N^{J-3}\,.$$

This result should be summed over all positions of the scalar impurity ϕ and finally over $J_1, \ldots J_4$. Clearly no relative phase will be associated when ϕ is in the first vertical block in figure 13. When it is in the second block, relative distance of ϕ and $\bar{\phi}$ to the interaction vertex is J_3, hence a nontrivial phase, q^{J_3} arises. The last case, when ϕ propagator is in the third block was already considered above and yields the phase q^{-J_2}. Replacing the sum over J_i (with $J_1 + J_2 + J_3 = J$) by an integral over $j_i = J_i/J$ (with $j_1 + j_2 + j_3 = 1$), one arrives at the integral

$$\int_0^1 \mathrm{d}j_1 dj_2 dj_3\, \delta(j_1+j_2+j_3-1)\left(j_2 e^{2\pi i n j_3} + j_3 e^{-2\pi i n j_2} + j_1\right)|1-e^{2\pi i n j_1}|^2 = \frac{1}{3}+\frac{5}{2\pi^2 n^2} \tag{5.4}$$

(for $n \neq 0$). Using the result for current correlator from appendix A, one finds the following D-term and external gluon contribution from the non-contractible diagrams,

$$\langle O^n_\mu(x)\bar{O}^m_\nu(y)\rangle \rightarrow \left(\frac{1}{3}+\frac{5}{2\pi^2 n^2}\right) G(x,y)^J \ln(\Lambda^2(x-y)^2)\frac{J_{\mu\nu}(x-y)}{(x-y)^2}\,. \tag{5.5}$$

The non-contractible F-term contribution in figure 13 arises when ϕ (and $\bar{\phi}$) impurity is at the first or last positions of the first block where one replaces the total vertex with an F-term quartic vertex. Note that the integral over this vertex gives the logarithmic scaling,

$$\frac{1}{(4\pi^2)^4}\int\frac{\mathrm{d}^4u}{(x-u)^4(y-u)^4}=2\pi^2\ln(\Lambda^2(x-y)^2)G(x,y)^2\,. \tag{5.6}$$

Now, one should dress this diagram by all possible locations of the derivatives. When both ∂_μ and ∂_ν hit the same line, the phase summation is equivalent to the situation discussed above. A double derivative line replaces the ϕ impurity whose position is to be summed over as in (5.4) and one again finds out the factor $(\frac{1}{3}+\frac{5}{2\pi^2n^2})$ together with the space-time dependence, $\partial_\mu\partial_\nu\frac{1}{(x-y)^2}$. The case where ∂_μ and ∂_ν hits different lines is handled in the same way as in section 2. One first considers a fixed position of ∂_μ, say l, and sum over position of ∂_ν from $l'=0$ to $J+1$ with the condition $l'\neq\sigma(l)$. This yields a factor $-q^l\bar{q}^{\sigma(l)}$ which is then summed over l and finally over $J_1,\ldots,J_4$ resulting in the same phase factor (5.4) up to a minus sign but with a different space-time dependence, $\partial_\mu\frac{1}{(x-y)^2}\partial_\nu\frac{1}{(x-y)^2}$. Combining these cases one gets,

$$\left(\frac{1}{3}+\frac{5}{2\pi^2n^2}\right)\frac{J_{\mu\nu}(x-y)}{(x-y)^2}$$

where we used (3.4). Therefore F-term contribution to anomalous part of the torus correlator is exactly the same as (5.5) and the total result involving D-term, external gluon and F-term contribution simply becomes,

$$\langle O^n_\mu(x)\bar{O}^m_\nu(y)\rangle_{\text{total}}\to\frac{\lambda' g_2^2}{4\pi^2}\left(\frac{1}{3}+\frac{5}{2\pi^2n^2}\right)G(x,y)^J\ln(\Lambda^2(x-y)^2)\frac{2J_{\mu\nu}(x-y)}{(x-y)^2}\,. \tag{5.7}$$

This torus dimension is exactly the same as torus anomalous dimension of scalar BMN operators, [9].

5.4 Special diagrams

All of the topological classes of Feynman graphs that have been discussed so far were available both for F-term and D-term parts of the lagrangian (4.1). However, the special Feynman graphs on the torus are formed when the interaction loops wind around different cycles and are present only if the interaction is a D-term quartic vertex (and their external gluon cousins). To see this, one should specify the orientation of the scalar propagator line $Z\bar{Z}$ (and $\phi\bar{\phi}$) by putting an arrow on it (not to be confused by derivatives). We choose the convention where scalar propagation is from O towards $\bar{O}$. With specification of the orientations, the F-term and D-term quartic vertices can be represented as in figure 16. One observes that the vertex where *adjacent lines have the opposite orientation* is only possible for D-terms. Using such a vertex one can draw 4 different special graphs on a torus. One of these possibilities is shown in figures 17 or 18. Here the shaded circle represents the total vertex, figure 2 as before. Special graphs can also be formed by external gluons as in figure 19. In general, special graphs are formed by combining either first or last lines of blocks 1 and 3 or blocks 2 and 4.

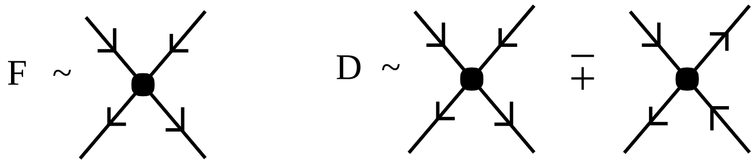


Figure 16: Orientations of F-term and D-term quartic vertices.

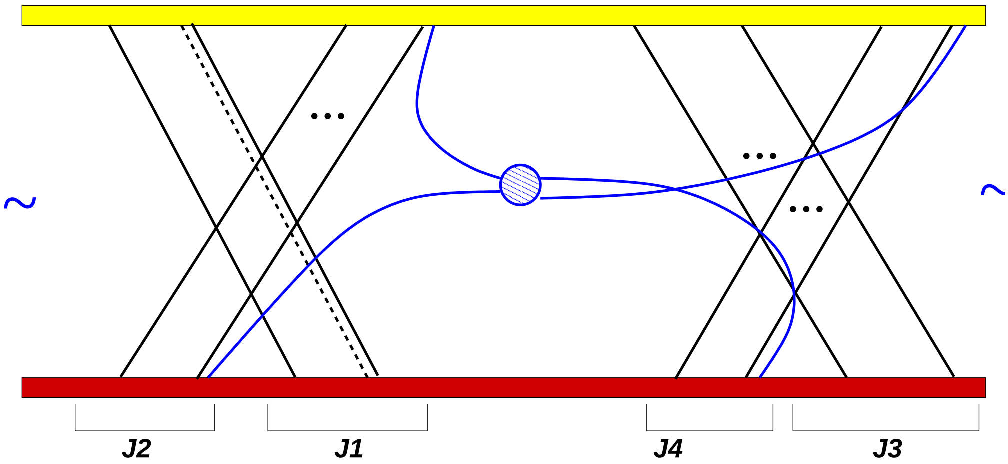


Figure 17: Special diagrams shown on a cylinder.

However, one makes a disturbing observation about special graphs: They are $\mathcal{O}(J^4)$ therefore all of the graphs we have considered so far are sub-leading with respect to them! Even worse, this extra power of J seems to be unsuppressed in the BMN limit, hence the presence of such graphs imply the breakdown of BMN perturbation theory!? Hopefully, as we shall demonstrate next, contribution of special graphs to the anomalous dimension is zero when one adds up all such possible graphs (figure 17) just as in the semi-contractible case.

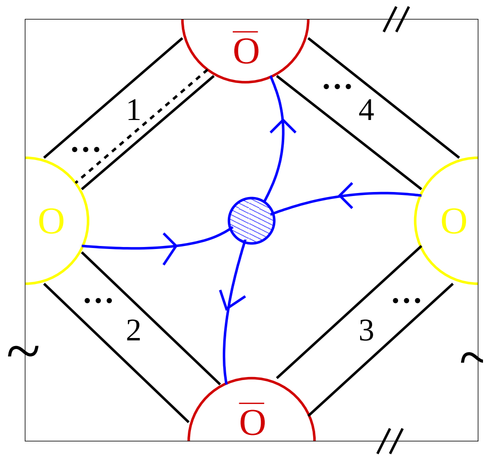


Figure 18: Periodic square representation of Special diagrams.

Let us consider a fixed position of ϕ at the last line of the first block and fixed $J_1, \ldots, J_4$. By use of trace identities given in appendix C and q-derivation tricks described above, one can easily boil down the special D-graphs into our familiar $\langle J\bar{J}\rangle$ correlator. Let us first consider the special contribution to the chiral primary correlator, $\mathcal{O}^J_\phi$. The trace identities show that figure 17

$$\sim (B+A)G^{J+2}N^{J+1}$$

hence special graph contribution to chiral primary correlator vanishes by non-renormalization theorem. Reader will find the details of this calculation in appendix B.

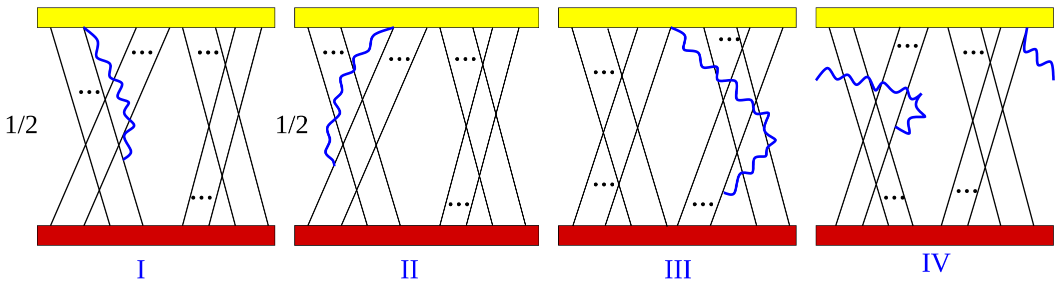


Figure 19: External gluons with special diagram topology.

Next we put in the q-derivatives on this graph to obtain the special contribution to $\langle O^n_\mu \overline{O}^m_\nu \rangle$ and observe that the only positions which yield a non-vanishing result is when both ∂_μ and ∂_ν act on the total vertex. Proof of this fact is exactly analogous to our argument in section 4.2. The algebraic tricks familiar from previous calculations are then used to express the result as

$$\sim q^{-J_2}\bar{r}^{J_1}(1-q^{-J_2-J_3})(1-\bar{r}^{-J_3-J_4})\langle J_\mu(x)\bar{J}_\nu(y)\rangle G^J N^{J+1} .$$

Similarly, the external gluon contributions shown in figure 19 can be shown to have the same form and total result — which follows from combining D-term, external gluon and self-energy contributions — becomes,

$$\sim q^{-J_2}\bar{r}^{J_1}(1-q^{-J_2-J_3})(1-\bar{r}^{-J_3-J_4})\langle U_\mu(x)\bar{U}_\nu(y)\rangle G^J N^{J+1} .$$

This was for the diagram in figure 17. A second special graph is obtained when the legs of the total vertex stretchs out into the last line of block 1 and last of block 3. Similarly a third graph is formed by first line in block 2 with first of 4 and a fourth graph by the first of 1 with first of 3. Let us now read off the phase factors of these four graphs respectively,

$$\begin{aligned} &(1-q^{-J_4-J_3})(1-\bar{r}^{-J_1-J_4})q^{-J_2}\bar{r}^{J_1}, \qquad &&(1-q^{-J_2-J_3})(1-\bar{r}^{-J_3-J_4}), \\ &(1-q^{-J_2-J_3})(1-\bar{r}^{-J_1-J_2})\bar{r}^{-J_3-J_4}, \qquad &&(1-q^{-J_2-J_1})(1-\bar{r}^{-J_1-J_4})q^{J_1}\bar{r}^{J_1}, \end{aligned}$$

respectively. Hence the contribution from 1st graph cancels out 4th graph and the 2nd cancels out th 3rd. *We conclude that contribution of special diagrams to both the vector anomalous dimension (the case $n=m$) and the operator mixing (the case $n \neq m$) vanishes* although it seems to be divergent as $J \to \infty$ at first sight. This shows that the only non-vanishing contribution is arising from non-contractible class of diagrams and the total correlator including $\mathcal{O}(\lambda')$ corrections both at the planar and the torus levels can now be written as,

$$\langle O^n_\mu(x)\bar{O}^n_\nu(y)\rangle = \Bigg\{(1+g_2^2 A_{nn})(1-n^2\lambda' \ln(\Lambda^2(x-y)^2) \\ + \frac{\lambda' g_2^2}{4\pi^2}\left(\frac{1}{3}+\frac{5}{2\pi^2 n^2}\right)\ln(\Lambda^2(x-y)^2)\Bigg\} G(x,y)^{J+2}\frac{2J_{\mu\nu}(x,y)}{(x-y)^2} . \tag{5.8}$$

This result clearly shows that the total contribution to the anomalous dimension of the vector type operator at $\mathcal{O}(g_{\rm YM}^2)$ and up to genus-2 level is exactly the same as the BMN anomalous dimension, that is,

$$\Delta = J + 2 + \lambda' n^2 - \frac{g_2^2 \lambda'}{4\pi^2}\left(\frac{1}{3} + \frac{5}{2\pi^2 n^2}\right). \tag{5.9}$$

As mentioned before, from the string theory point of view, the torus anomalous dimension is identified with the genus-one mass renormalization of the corresponding state.

Let us briefly describe how the non-vanishing of torus level anomalous dimension implies the non-vanishing of the $\mathcal{O}(\lambda')$ interacting three-point functions,

$$\left\langle \bar{O}_\mu^{n,\,J} O_\nu^{m,\,J'} O^{J-J'} \right\rangle \qquad \text{and} \qquad \left\langle \bar{O}_\mu^{n,\,J} O_\nu^{J'} O_\phi^{J-J'} \right\rangle$$

through the unitarity sum. Here the relevant supergravity operators are defined in (2.3) and (2.4). As mentioned in the introduction, this is puzzling since string field theory result of [15] shows that r.h.s. of (1.9) vanishes for vector operators.

It was argued in [9] that one can handle the string interactions effectively with non-degenerate perturbation theory of a quantum mechanical system. Unitarity sum gives the following 2nd order shift in the energy of the string state with momentum n,

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle i'|P^-|j'k'\rangle|^2}{E_n^{(0)} - E_m^{(0)}}\,. \tag{5.10}$$

Here $|i'\rangle$ is the string excitation with momentum n and $|j'k'\rangle$ represents all possible intermediate states with momentum m. In the case of $|i'\rangle = \alpha_n^{\phi\,\dagger}\alpha_{-n}^{\mu\,\dagger}|0,p^+\rangle$ which is the dual of vector operator (2.10), there are two possibilities for the intermediate states:

1. $|j'\rangle = \alpha_m^{\phi\,\dagger}\alpha_{-m}^{\mu\,\dagger}|0,p_1^+\rangle$ and $|k'\rangle = |0,p_2^+\rangle$ with $p_1^+ + p_2^+ = p^+$. Corresponding operators are, $O_\nu^{m,\,J'}$ and $O^{J-J'}$ respectively.

2. $|j'\rangle = \alpha_0^{\phi\,\dagger}|0,p_1^+\rangle$ and $|k'\rangle = \alpha_0^{\mu\,\dagger}|0,p_2^+\rangle$. Corresponding operators are the BPS operators, $O_\nu^{J'}$ and $O_\phi^{J-J'}$.

Therefore the sum in (5.10) involves a sum over these two cases together with sub-summations over m and J'. Vanishing of $\langle i'|P^-|j'k'\rangle$ for both of the cases above implies that $E_n^{(2)} = (\Delta - J)_{torus} = 0$ *for the vector operator*. A loophole in this argument is that we only considered the cubic string vertex in the effective description whereas the *contact terms* may also contribute the mass renormalization of the string states hence give rise to a non-zero torus level anomalous dimension in the dual theory. We come back to this issue in the last section.

6. A SUSY argument

The fact that BMN and vector operators (which belong to separate SO(4) sectors of the gauge theory) have equal anomalous dimensions both at planar and torus levels suggests

	ϕ	ψ	Z	A_μ	θ_ϕ	θ_ψ	θ_Z	λ	Q^1	Q^2	Q^3	Q^4
J_{12}	1	0	0	0	1/2	−1/2	−1/2	1/2	−1/2	1/2	1/2	−1/2
J_{34}	0	1	0	0	−1/2	1/2	−1/2	1/2	1/2	−1/2	1/2	−1/2
J_{56}	0	0	1	0	−1/2	−1/2	1/2	1/2	1/2	1/2	−1/2	−1/2

Table 1: R-charges of component fields and SUSY generators.

that there might be a $\mathcal{N} = 4$ SUSY transformation relating these two operators. Whereas the equality of the planar anomalous dimensions of these operators is required by the consistency of BMN conjecture, there is no *a priori* reason to believe that this equality persists at higher genera. A SUSY map, however, would protect $\Delta_{BMN} - \Delta_{vector} = -1$ at all loop orders and all genera.

In this section, we will see that indeed there is such a transformation which maps the BMN operator onto vector operator plus a correction term. We will argue that the correction is negligible in the BMN limit and hence expect the equality of anomalous dimensions, *both at planar and torus levels.*

The supersymmtery transformations of $\mathcal{N} = 4$ SYM has recently been derived in [22]. In SU(4) symmetric notation, the transformations of the scalars and chiral spinors read,

$$\delta_\epsilon X^{AB} = -i\left(-\bar{\epsilon}_-^A \theta_+^B + \bar{\epsilon}_-^B \theta_+^A + \epsilon^{ABCD}\theta_{+C}\epsilon_{-D}\right) \tag{6.1}$$

$$\delta_\epsilon \theta_+^A = \frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon_+^A + 2D_\mu X^{AB}\gamma^\mu \epsilon_{-B} + 2i[X^{AC}, X_{CB}]\epsilon_+^B \,. \tag{6.2}$$

in which $A = 1, \ldots 4$ is an SU(4) index and $X^{AB} = -X^{BA}$.

We will use these transformation rules in a somewhat schematic way, since the information we need can be obtained more simply by classifying all fields and supercharges with respect to the decomposition $\mathrm{SU}(4) \to \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$. The three commuting U(1) charges can be viewed as J_{12}, J_{34} and J_{56} in SO(6). All fermionic quantities are taken as 2 component Weyl spinors. The four spinor fields are denoted by θ_ϕ, θ_ψ, θ_Z, $\theta_{A_\mu} = \lambda$ where the subscript indicates their bosonic partner in an $\mathcal{N} = 1$ decomposition of $\mathcal{N} = 4$ SUSY. The fermionic transformation rule above may be interpreted as,

$$\{\overline{Q}_{+B}, \theta^A\} \sim F_{\mu\nu}\gamma^{\mu\nu}\delta_A^B + \cdots \,,$$

showing that $\overline{Q}_{+A}$ has the same U(1) quantum numbers as θ^A. In general fermions and anti-fermions have opposite U(1) charges, as in the case of conjugate bosons. The product of these 3 charges is positive on the $\overline{Q}_{+A}$. With these remarks in view, we can write table 1 of U(1) charges.

We now apply the transformation rules (6.1) and (6.2) in the $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ basis in which all transformations which conserve the U(1) charges are allowed. Consider the action of Q_α^2 on the BMN operator (2.6). We see that ϕ and Z's are left unchanged whereas ψ is transformed into a gaugino λ, i.e.

$$[Q^2, O_{\phi\psi}^n] \propto \sum_{l=0}^{J} e^{\frac{2\pi i n l}{J}}\, \mathrm{Tr}(\phi Z^l \lambda Z^{J-l}) \,.$$

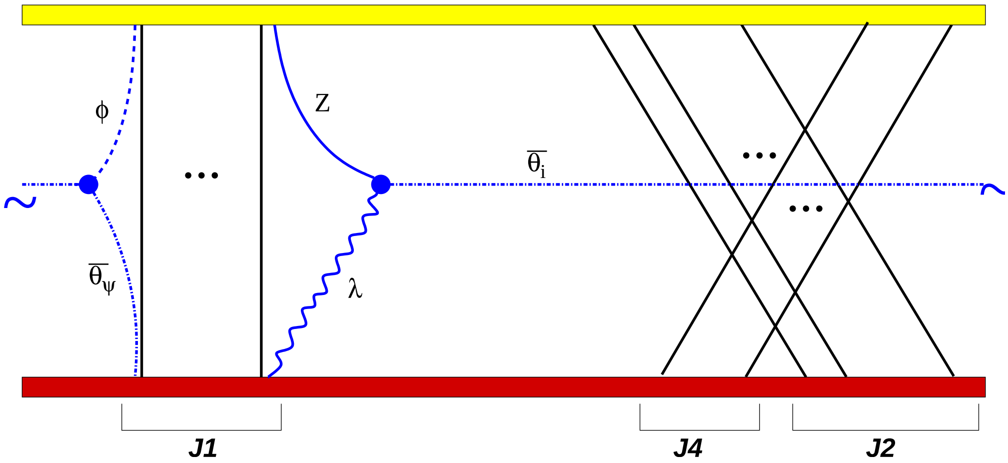


Figure 20: Torus level non-contractible contribution to $\langle O^n_\mu \bar{O}^m_f \rangle$ transition amplitude. Derivative in O^n_μ can be placed on any line connected to O^n_μ, although it is not shown explicitly. There is a similar graph obtained by interchanging internal vertices.

Next we act on this with another anti-chiral supercharge $\overline{Q}^3_\alpha$ with quantum numbers $(-1/2, -1/2, +1/2)$. According to table 1 and transformation rules given in (6.2), Z's again remain unchanged, λ is transformed into $D_\mu Z$[6] and ϕ is transformed into $\bar{\theta}_\psi$, i.e.

$$\{\overline{Q}^3, [Q^2, O^n_{\phi\psi}]\} \propto \sum_{l=0}^{J} e^{\frac{2\pi i n l}{J}} \,\mathrm{Tr}\left(\phi Z^l (D_\mu Z) Z^{J-l}\right) + \\ + \sum_{l=0}^{J} e^{\frac{2\pi i n l}{J}} \,\mathrm{Tr}\left(\bar{\theta}_\psi Z^l \lambda Z^{J-l}\right) \equiv \widetilde{O}^n_\mu + O^n_f \,. \tag{6.3}$$

Therefore supersymmetry guarantees that $\widetilde{O}^n_\mu + O^n_f$ has the same $\Delta - J$ with the BMN operator.

Note that the first term is not quite the same as the vector operator of (2.10) but there are two differences. However, for large but finite J, the difference between contributions of $\widetilde{O}^n_\mu$ and O^n_μ to the correlators is $\mathcal{O}(1/J)$. This is because the exceptional piece in (2.10) where D_μ is acting on the impurity ϕ is $\mathcal{O}(1/J)$ with respect to the first term of (2.10) hence negligible in the dilute approximation. Secondly, the difference between the definitions of q for $\widetilde{O}^n_\mu$ and O^n_μ, i.e. $q^J = 1$ and $q^{J+2} = 1$ respectively, is also $\mathcal{O}(1/J)$.

Now consider computing the dimension of $\widetilde{O}^n_\mu + O^n_f$ at the planar level. This would be the same as the dimension of only $\widetilde{O}^n_\mu$ provided that the transition amplitude $\langle \widetilde{O}^n_\mu \bar{O}^m_f \rangle$ is negligible. Let us first consider the correlator $\langle O^n_\mu \bar{O}^m_f \rangle$ instead. Above we explained that the difference between the contributions of $\widetilde{O}^n_\mu$ and O^n_μ to the anomalous dimension is $\mathcal{O}(1/J)$, therefore conclusions made for $\langle O^n_\mu \bar{O}^m_f \rangle$ will also be valid for $\langle \widetilde{O}^n_\mu \bar{O}^m_f \rangle$ as $J \to \infty$ in the BMN limit. The leading contribution to this transition amplitude in $\mathcal{O}(\lambda')$ arises from Z-λ-$\bar{\theta}$ (5th term in (4.1)) and the Yukawa interaction (6th term in (4.1)): One of the

[6] The last term in (6.2) which is quadratic in the scalars does not give correct quantum numbers for $J_{12} \ldots J_{56}$ hence is not present.

Z's in O_μ^n splits into a θ_Z and $\bar\lambda$ and θ_Z gets absorbed by ϕ turning into $\bar\theta_\psi$ through the Yukawa interaction. See figure 20 for the analogous interaction at the torus level. Note that contribution to the *planar* dimension requires the Z that is taking place in the interaction and ϕ be adjacent. Note also that the derivative in O_μ^n can be at any position. Before acting with the q-derivative, the integration over the internal vertices together with the scalar propagators yields,

$$\sim \ln((x-y)^2\Lambda^2)G^J \,.$$

Since the anomalous dimension is the coefficient of the log term, anomalous contributions arise when the ∂_μ act on G's but not on the log. Therefore any position of the derivative in O_μ^n in the planar diagram (also any position of the derivative in figure 20 in torus case) gives the same contribution,

$$\sim \ln((x-y)^2\Lambda^2)G^{J+1} \,,$$

regardless it is acting on the fields participating in the interaction or not. Then the phase sum over the position of ∂_μ gives (using $q^{J+2}=1$),

$$\sum_{l=0}^{J+1} q^l = 0 \,.$$

Therefore the transition amplitude $\langle O_\mu^n \bar{O}_f^m\rangle_{\text{planar}}$ vanishes identically! As we described above this implies that, for large but finite J, $\langle \widetilde{O}_\mu^n \bar{O}_f^m\rangle_{\text{planar}}$ do not vanish but supressed with a factor of $1/J$ with respect to $\langle \widetilde{O}_\mu^n \bar{\widetilde{O}}_\mu^n\rangle_{\text{planar}}$ in the BMN limit. Therefore we see that *supersymmetry together with large J suppression is capable to explain why vector and BMN anomalous dimensions are equal at the planar level.*

This argument can easily be extended to the torus level. In section 5.4 it was proven that the only torus level contribution for the vector correlator $\langle O_\mu^n \bar{O}_\nu^m\rangle_{torus}$ comes from the non-contractible diagram, figure 13. Recall that this diagram is of $\mathcal{O}(J^3)$ because there are three blocks of Z lines and no phase suppression (unlike contractible or semi-contractible cases). Including the $1/(JN^2)$ normalization factor we found out that the diagram is of $\mathcal{O}(g_2^2)$. We want to see how the transition amplitude at the torus level, $\langle \widetilde{O}_\mu^n \bar{O}_f^m\rangle_{torus}$ goes with J. Instead, let us again consider the correlator $\langle O_\mu^n \bar{O}_f^m\rangle_{torus}$. One can easily see that (with the same argument presented in the beginning of section 5) all possible torus diagrams of $\langle O_\mu^n \bar{O}_f^m\rangle$ can be divided into four seperate classes of section 5. Let us consider the non-contractible diagram for example. The diagram is shown in figure 20. The derivative in O_μ^n can be at any position. Hence the phase summation over the position of the derivative vanishes identically just as in the planar case. For the same reason the external gluon contribution vanishes as well (together with other torus diagrams: contractible, semi-contractible and special). One concludes that for large but finite J, $\langle \widetilde{O}_\mu^n \bar{O}_f^m\rangle_{torus}$ is again $1/J$ suppressed with respect to $\langle O_\mu^n \bar{O}_\nu^m\rangle_{torus}$. We see that *supersymmetry in the BMN limit, is also capable to explain the equality of vector and BMN anomalous dimensions at the torus level.* However we emphasize that this equality is not exact but only holds in large J limit. Therefore, whether this reasoning can be extended to higher orders in genus remains as an interesting question.

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As an aside we state another important conclusion. The fact that the transition amplitude is negligible with respect to $\langle O^n_\mu \bar{O}^m_\nu \rangle$ shows that *the vector operator* O^n_μ *and the fermionic impurity BMN operator* O^n_f *has the same planar and torus anomalous dimensions*. This gives an easy method to generate all BMN operators which carry the same anomalous dimension as the scalar operator by acting on it with the supercharges Q^I arbitrary times *and* making sure that the transition amplitudes among all of the pieces in the end-product is negligible in the BMN limit.

7. Discussion and outlook

In this paper we computed the two-point function of vector impurity type BMN operator at planar and torus levels, for small λ' (large $\mu p^+ \alpha'$). In this regime, SYM is weakly coupled and we only considered interactions at $\mathcal{O}(\lambda')$ order in SYM interactions. Our result for the total anomalous dimension is given in eq. (5.9). This turns out to be exactly the same as scalar impurity type anomalous dimension which was computed in [9] *both at planar and torus levels*. This result provides two tests on the recent conjectures. This equality at the planar level constitutes a non-trivial check on the BMN conjecture. Secondly, the non-zero torus anomalous dimension is a field theory prediction which should match the string theory result for the mass renormalization of the vector states. We mentioned at the end of section 5 however that this non-zero torus dimension raises a puzzle since the string field theory cubic vertex for vector the string states vanishes [15]. Our results are further supported by the SUSY argument given in the previous section.

We would like to briefly address some possible resolutions of this contradiction between string field theory and gauge theory results. Generally speaking, there is another type of interaction in light-cone string field theory [26] apart from the cubic string vertex. This arises from the *contact terms* and was not taken into account in the calculation of [15]. In the context of IIB strings in pp-wave background this issue was discussed in [21]. Contact terms arise from a quartic string vertex whose presence is required by supersymmetry [24, 23]. Contribution of contact terms to mass renormalization is $\mathcal{O}({g'_s}^2)$ and there seems no *a priori* reason to ignore it. In case these terms are indeed non-negligible they might give rise to a non-zero torus anomalous dimension in the dual field theory.

Another resolution[7] of the gauge/string contradiction would be that peturbative gauge theory calculations for the interacting three-point function are not capable to probe the short distance ($< 1/\mu$) physics on the world-sheet. Recall that [15] it is the *prefactor* of cubic vertex which suggests the vanishing of $\langle i'|P^-|j'\rangle|k'\rangle$ in case of the vector state: Spradlin and Volovich have pointed out that the short distance limit on the world-sheet and the weak gauge coupling limit ($\mu \to \infty$) do not commute. To be able to obtain the *prefactor* one should first take the short distance limit. Then one takes large μ limit to obtain an expression for the weakly coupled three-point function. This procedure expects vanishing of $\langle i'|P^-|j'\rangle|k'\rangle$. On the other hand, exchanging the limits, hence loosing the contribution

[7] I am Grateful to L. Motl for mentioning this idea to me.

of *prefactor* would suggest non-zero interacting three-point function also for the vector operator. It is a possibility that perturbative SYM is not able to "discover" the *prefactor* of string field theory but able to see only a $1/\mu$ expansion of the delta-functional. This would be another line of reasoning to explain why our perturbative calculation produced a non-zero torus anomalous dimension for the vector operator. Clearly, a perfect understanding of the map between weakly coupled string/gauge theories should resolve this apparent contradiction.

A number of directions for further study of these issues are the following. One can take a direct approach and compute the contact terms in string field theory to compare its contribution to torus level mass renormalization with the contribution of non-contractible diagrams to the torus anomalous dimension. A similar strategy in gauge theory side is to obtain the interacting three-point function of the vector operators as a direct test of (1.9). It is also desirable to go beyond $\mathcal{O}(\lambda')$ and obtain a non-perturbative formula for the anomalous dimension of the vector operators with a similar calculation as in [1] for the case of scalar operators.

Acknowledgments

It is a pleasure to thank David Berenstein, Matt Headrick, Shiraz Minwalla, Juan Maldacena, Horatiu Nastase and Witek Skiba for useful discussions. I must specially thank to Dan Freedman and Lubos Motl for their early participation and constant help during the preperation of the paper. Section 6 is worked out together with them. This work is supported by funds provided by the D.O.E. under cooperative research agreement #DF-FC02-94ER40818.

A. D-term and external gluon contributions at $\mathcal{O}(\lambda')$

The aim of this appendix is to prove equation (4.11) showing the planar level contribution to the vector anomalous dimension which arise from the D-term part of the lagrangian, (4.1). In section 3 we explained how one can express this result in terms of the the correlator of a non-conserved current, $J_\mu = Z \overset{\leftrightarrow}{\partial} Z$. Here, we shall compute D-term contributions to this correlator at the one loop level. In the following we will first show that D-term quartic vertex do not contribute to $\langle J\bar{J}\rangle$ at all. Then we will explain how to compute the self-energy contributions. Finally we shall consider the contribution arising from the gluon exchange graph figure 4I by employing the trick of relating it to simpler diagrams figures 4II, 4III, 4IV as we described at the end of section 4.2.

The most direct and painless way to compute these Feynman diagrams is the beautiful method of differential renormalization (DR) [19]. The main idea is to compute n-point functions $\langle O_1(x_1)\cdots O_k(x_k)\rangle$ directly in space time rather than Fourier transforming to momentum space, and adopting a certain differential regularization scheme when the space-time expressions become singular, i.e. as $x_i \to x_j$. Note that, away from the contact points $x_i \to x_j$, the n-point function is well-defined and can be Fourier transformed back to momentum space. However as x_i approachs to x_j for $i \neq j$, most expressions become too

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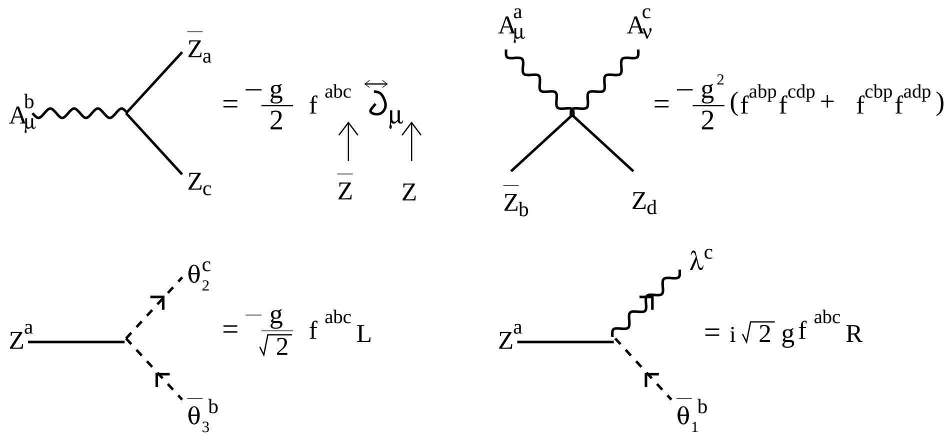

Figure 21: Feynman rules for vertices. Same rules hold when Z is replaced by ϕ. In the θ-Z-$\bar{\theta}$ vertex, exchanging chiral fermion flavor 2 with 3 gives a minus sign and replacing Z with $\bar{Z}$ changes the chirality projector from L to R. The analogous $\bar{Z}$-$\bar{\lambda}$-θ vertex is obtained by replacing R with $-L$.

singular to admit a Fourier transform. Yet, one can easily rewrite the singular expressions in terms of derivatives of less singular expressions hence render the Fourier transform possible. The only such rewriting we will use here is the following formula [19],

$$\frac{1}{(x_1 - x_2)^2} = -\frac{1}{4}\Box \frac{\ln((x_1 - x_2)^2\Lambda^2)}{(x_1 - x_2)^2} \tag{A.1}$$

where Λ^2 is the renormalization scale. We also remind the Green's equation in our conventions,

$$\Box \frac{1}{(x_1 - x_2)^2} = -4\pi^2 \delta(x_1 - x_2)\,. \tag{A.2}$$

A nice feature of DR is that one can adopt a renormalization scheme where one ignores all tadpole diagrams, simply by setting them to zero. We will work with the euclidean signature throughout the appendices in which case the space-time Feynman rules for $\mathcal{N} = 4$ SYM are read off from (4.1). Scalar, gluon and fermion propagtors read

$$\frac{\delta_{ab}}{4\pi^2(x-y)^2}\,; \qquad \frac{\delta_{ab}\delta_{\mu\nu}}{4\pi^2(x-y)^2}\,; \qquad \frac{\delta_{ab}}{4\pi^2}\not{\partial}_x \frac{1}{(x-y)^2}\,.$$

The interaction vertices are shown in figure 21.

It is very easy to see that D-term contribution vanishes.

$$\begin{aligned}\langle \mathrm{Tr}(J_\mu J_\nu)\rangle_D = {} & \frac{g_{\mathrm{YM}}^2}{4\pi^2}(f^{abd}f^{ace} + f^{abe}f^{acd})\,\mathrm{Tr}(T^aT^bT^cT^d)\times \\ & \times \int \mathrm{d}^4u \left(\frac{1}{(x-u)^2}\overleftrightarrow{\partial}_\mu \frac{1}{(y-u)^2}\overleftrightarrow{\partial}_\nu \frac{1}{(y-u)^2}\right) = 0\,.\end{aligned}$$

To find the self-energy contribution to $\langle J\bar{J}\rangle$ let us first compute self-energy corrections to scalar propagator. These arise from three sources: a gluon emission and reabsorbtion, chiral-chiral fermion loop and chiral-gluino fermion loop. We will not need the exact value of the first contribution as will be explained below. Let us begin with chiral-criral loop which is shown in figure 23. Calling this graph SE_1, Feynman rules yield,

$$
\begin{aligned}
SE_1 &= -2\frac{1}{4\pi^2}f^{acd}\left(-\frac{g}{\sqrt{2}}\right)f^{bdc}\left(\frac{g}{\sqrt{2}}\right)\times \\
&\quad\times\int \mathrm{d}^4u\mathrm{d}^4v\frac{1}{(x-u)^2}\frac{1}{(y-v)^2}\,\mathrm{Tr}\left[L\not{\partial}_u\frac{1}{(u-v)^2}R\not{\partial}_v\frac{1}{(u-v)^2}\right] \\
&= -2\frac{g_{\mathrm{YM}}^2\delta^{ab}N}{4\pi^2}\int \mathrm{d}^4u\mathrm{d}^4v\frac{1}{(x-u)^2}\frac{1}{(y-v)^2}\not{\partial}^u_\alpha\frac{1}{(u-v)^2}\not{\partial}^v_\alpha\frac{1}{(u-v)^2}
\end{aligned}
$$

where factor of 2 in the first line comes from summing over two fermion flavors θ_1 and θ_2. We will show the evaluation of integral here for future reference. By parts in u gives,

$$
\begin{aligned}
I_1(x,y) &= \int \mathrm{d}^4u\mathrm{d}^4v\left\{\frac{1}{(x-u)^2}\frac{1}{(y-v)^2}(-4\pi^2)\frac{\delta(u-v)}{(u-v)^2}+\frac{1}{2}\frac{1}{(x-u)^2}\frac{1}{(y-v)^2}\Box\frac{1}{(u-v)^4}\right\} \\
&\to -\frac{1}{8}\int \mathrm{d}^4u\mathrm{d}^4v\frac{1}{(x-u)^2}\frac{1}{(y-v)^2}\Box_u\Box_v\frac{\ln((u-v)^2\Lambda^2)}{(u-v)^2} \\
&= -\frac{1}{8}(-4\pi^2)^2\int \mathrm{d}^4u\mathrm{d}^4v\delta(x-u)\delta(y-v)\frac{\ln((u-v)^2\Lambda^2)}{(u-v)^2} \\
&= -\frac{(4\pi^2)^2}{8}\frac{\ln((x-y)^2\Lambda^2)}{(x-y)^2} \qquad\qquad (A.3)
\end{aligned}
$$

In passing to second line we omitted the first term which is supposed to cancel out with tadpole contributions in DR. Second and third equalities use (A.1) and (A.2). Hence,

$$
SE_1 = \frac{g_{\mathrm{YM}}^2N}{4(4\pi^2)^2}\delta_{ab}\frac{\ln((x-y)^2\Lambda^2)}{(x-y)^2}\,. \qquad (A.4)
$$

An analogous computation for the gluino-chiral fermion loop gives,

$$
SE_2 = -\frac{g_{\mathrm{YM}}^2N}{2(4\pi^2)^2}\delta_{ab}\frac{\ln((x-y)^2\Lambda^2)}{(x-y)^2}\,. \qquad (A.5)
$$

Although we will not need it, let us give here the total self-energy correction to the scalar propagator for reference (including gluon emission-reabsorbtion),

$$
SE = -\frac{g_{\mathrm{YM}}^2N}{8(4\pi^2)^2}\delta_{ab}\frac{\ln((x-y)^2\Lambda^2)}{(x-y)^2}\,. \qquad (A.6)
$$

Turning to gluon exchange contribution to $\langle J\bar{J}\rangle$, we recall our trick to express it as gluon exchange correction to gluon propagator in scalar QED, figure 4III,

$$
\begin{aligned}
\langle \mathrm{Tr}(J_\mu(x)\bar{J}_\nu(y))\rangle_{g.e.} &= \frac{\delta_{ab}}{2}\langle J^a_\mu(x)\bar{J}^b_\nu(y))\rangle_{g.e.} \\
&= -\frac{N^4}{g_{\mathrm{YM}}^2}\langle A_\mu(x)A_\nu(y)\rangle_{g.e.} \qquad (A.7)
\end{aligned}
$$

where $J^a_\mu = if^{abc}Z^b\partial_\mu Z^c$. In the second line we divided out by a factor of $(g/2)^2$ to compensate for the coupling of incoming and outgoing gluons to the loop and there is an overall -1 w.r.t. $\langle J\bar{J}\rangle$ because of the antisymmetric derivative in scalar-gluon vertex. We also took into account the color factors at four vertices,

$$f^{cge}f^{dac}f^{fdg}f^{ehf}=\delta^{ab}\frac{N^2}{2}\,.$$

Now, the sub-divergent piece of this diagram cancels out the sub-divergent pieces of graphs I and II in figure 4. Hence the contribution to anomalous dimension is,

$$\langle A_\mu(x)A_\nu(y)\rangle_{g.e.}\to -4\times\text{figure 4I}-2\times\text{figure 4II}\,.$$

When we include the self-energy corrections to $\langle J\bar{J}\rangle$ second term will be canceled out by gluon emission-reabsorbsion part of the self energies and one is left with,

$$\langle \mathrm{Tr}(J_\mu\bar{J}_\nu)\rangle_{g.e.+s.e.}=-\frac{N^4}{g_{\rm YM}^2}\{-4\times\text{figure 4I}-2\times F.S.E.\}\tag{A.8}$$

with "FSE" being only the fermion loop contributions to the self energy of $\langle J\bar{J}\rangle$,

$$FSE=-\frac{g_{\rm YM}^2N^4}{16\pi^2}\frac{J_{\mu\nu}}{(x-y)^2}\ln((x-y)^2\Lambda^2)G(x,y)^2\,.\tag{A.9}$$

Note that we included 1/2 factor coming from our counting of self energies, (see figure 2). Arousal of conformal factor, $J_{\mu\nu}$, is explained below. Let us now compute the contribution of figure 4I including the color factors in conversion to $\langle J\bar{J}\rangle$. Using the Feynman rules for in figur 21,

$$\begin{aligned}\text{Figure 4a}\to&-\frac{g_{\rm YM}^2}{2}(f^{acp}f^{dhp}+f^{dcp}f^{ahp})\left(-\frac{g}{2}f^{edh}\right)\left(-\frac{g}{2}f^{cbe}\right)\times\\&\times\frac{1}{(4\pi^2)^4}\int\frac{d^4u}{(x-u)^2}\left(\frac{1}{(x-y)^2}\overset{\leftrightarrow y}{\partial_\nu}\frac{1}{(y-u)^2}\overset{\leftrightarrow u}{\partial_\mu}\frac{1}{(x-u)^2}\right)\\&=-N^2\delta^{ab}\frac{9}{32}\frac{g^4}{(4\pi^2)^4}\frac{1}{(x-y)^2}\overset{\leftrightarrow}{\partial_\nu}\,\partial_\mu I_2(x,y)\end{aligned}$$

where the integral is

$$I_2(x,y)=\int\frac{d^4u}{(x-u)^4}\frac{1}{(y-u)^2}=\pi^2\frac{\ln((x-y)^2\Lambda^2)}{(x-y)^2}\tag{A.10}$$

again by use of (A.1) and (A.2). The anomalous contribution is obtained by keeping terms proportional to ln in (A.10) which gives,

$$\text{Figure4a}\to-\frac{9}{64}\frac{g^4}{4\pi^2}\delta^{ab}N^2\ln((x-y)^2\Lambda^2)\frac{J_{\mu\nu}(x,y)}{(x-y)^2}G(x,y)^2\,.$$

Putting this in (A.8) together with (A.9) one gets,

$$\begin{aligned}\langle \mathrm{Tr}(J_\mu\bar{J}_\nu)\rangle_{g.e.+s.e.}&=-\frac{2}{g_{\rm YM}^2}\delta_{ab}\left(4\times\frac{9}{64}-\frac{1}{4}\right)\delta^{ab}\frac{g_{\rm YM}^2}{4\pi^2}N^2\ln((x-y)^2\Lambda^2)\frac{J_{\mu\nu}(x,y)}{(x-y)^2}G(x,y)^2\\&=-\frac{5g_{\rm YM}^2N}{4\pi^2}(\frac{N}{2})^3\ln((x-y)^2\Lambda^2)\frac{J_{\mu\nu}(x,y)}{(x-y)^2}G(x,y)^2.\end{aligned}\tag{A.11}$$

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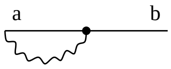
This is the total contribution to $\langle J\bar{J}\rangle$ from D-term, gluon exchange and self energies. Insertion of this result into (4.10) yields the desired result, (4.11).

Our next task is to fill in the details in the computation of section 4.3 that leads to the contribution of external gluons to the $\mathcal{O}(\lambda')$ anomalous dimension, (4.19). These contributions are shown in figure 6. To evaluate Graph I and II of figure 6, we will need the function C_μ which was defined in (4.13). Recall that external gluon is coming from the commutator $ig[A_\mu, Z]$ which contributes $-gf^{acd}$ at the external vertex where c is associated with the gluon, d with Z-line and a is the color factor of the external vertex. Use of the Feynman rule in figure 21 for the internal vertex gives,



$$\begin{aligned}
&= (-gf^{acd})(-\frac{g}{2}f^{dcb}\frac{1}{(4\pi^2)^3}\int\frac{\mathrm{d}^4u}{(x-u)^2}\left(\frac{1}{(x-u)^2}\overleftrightarrow{\partial}_{u\,\mu}\frac{1}{(y-u)^2}\right) \\
&= -\frac{g_{\mathrm{YM}}^2}{2}N\delta^{ab}\frac{1}{(4\pi^2)^3}(-\partial_\mu^y+\frac{1}{2}\partial_\mu^x)I_2(x,y) \\
&= \frac{3}{16}\frac{g_{\mathrm{YM}}^2 N}{4\pi^2}\delta^{ab}\partial_\mu^y\left(\ln((x-y)^2\Lambda^2)G(x,y)\right)
\end{aligned}$$

Hence we read off

$$C_\mu = \frac{3}{16}\frac{g_{\mathrm{YM}}^2}{4\pi^2}\partial_\mu^y\left(\ln((x-y)^2\Lambda^2)G(x,y)\right).$$

Inserting this into (4.14) yields the total contribution to $\langle O_\mu^n\bar{O}_\nu^m\rangle$ from Graph I,

$$\left(\frac{N}{2}\right)^{J+2}(J+2)\delta_{nm}\frac{3}{16}\frac{g_{\mathrm{YM}}^2 N}{4\pi^2}G(x,y)^J G(x,y)\overleftrightarrow{\partial}_\nu\,\partial_\mu^y\left(\ln((x-y)^2\Lambda^2)G(x,y)\right).$$

We add to this the contribution of Graph II which is identical except a phase factor of $q\bar{r}$ and their horizontal reflections which doubles the total answer. Anomalous contribution is obtained by keeping terms proportional to log which is (4.15) after correctly normalizing according to (2.10). Contributions of Graph III and IV (and their horizontal reflections) are identical to above except one the factor $1+q\bar{r}$ is replaced by $-q-\bar{r}$, hence the final answer is (4.19).

B. Trace identities and F-term contribution at $\mathcal{O}(\lambda')$

Here, we first present the trace identities which were used throughout the calculations and work out an example to show how to use them in calculations of n-point functions. The example we choose is the special diagram — section 5.4 — which arises in D-term contribution to torus two-point functions. As another application of the trace identities we compute the F-term contribution to anomalous dimension of vector operators.

Let us fix the convention by,

$$\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$$

and trivially extend SU(N) structure constants, f^{abc}, to U(N) by adding the $N\times N$ matrix $T^0 = \frac{I}{\sqrt{2N}}$. For notational simplicity let us denote all explicit generators by their index values, i.e. $T^a \to a$ and replace explicit trace of an arbitrary matrix M by $\mathrm{Tr}(M) \to (M)$.

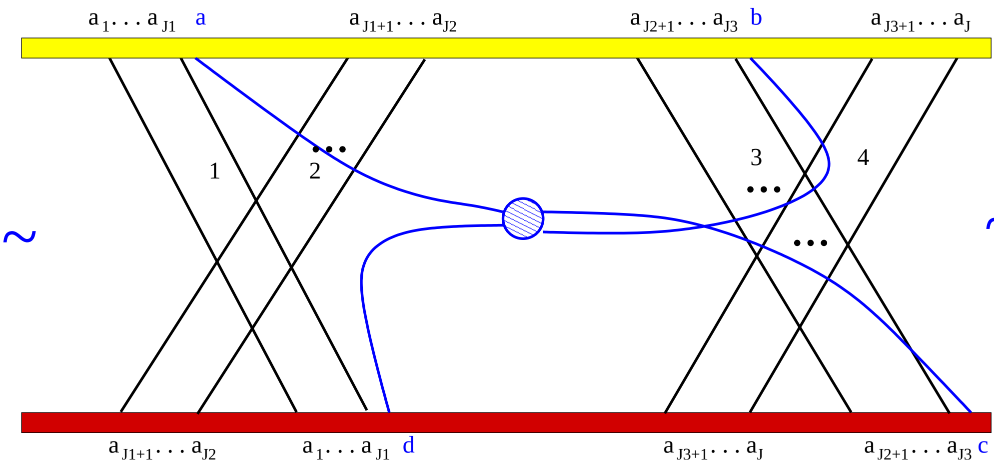


Figure 22: A torus level diagram with "special" topology.

Results derived in [25] can be used to prove the following trace identities.

$$
\begin{aligned}
(MaM'a) &= \frac{1}{2}(M)(M') \qquad (ab) = \frac{1}{2}\delta^{ab} \\
(Ma)(aM') &= \frac{1}{2}(MM') \qquad (a) = \sqrt{\frac{N}{2}}\delta^{a0} \\
aa &= \frac{1}{2}NI \qquad (\,) = N
\end{aligned} \tag{B.1}
$$

Let us use these identities on the example of figure 22. This shows a D-term interaction in a special diagram. We would like to show that the total trace involved in this graph boils down to a trace over the four interacting legs and eventually to show that this diagram is indeed at torus level by algebraic methods. In figure 22, the color indices carried by the block of Z-lines are $a_1, \cdots a_{J_1}$, $a_{J_1+1}, \cdots a_{J_2}$ $a_{J_2+1}, \cdots a_{J_3}$ and $a_{J_3+1}, \cdots a_J$ respectively. Color indices of the interacting lines are denoted as a, b, c, d. Then the color factor associated with the vertex is $f^{acp}f^{dbp}$. Interestingly, untwisting the b and c lines in figure 22 which give the other color combination, $f^{abp}f^{dcp}$ turns out to be a genus-2 diagram! That's why we draw the special and semi-contractible diagrams of section 4 with twists. Use of (B.1) in figure 22 goes as follows,

$$
\begin{aligned}
&(a_1 \cdots a_{J_1}\, a\, a_{J_1+1} \cdots a_{J_2} a_{J_2+1} \cdots a_{J_3}\, b\, a_{J_3+1} \cdots a_J) \times \\
&\times (c\, a_{J_3} \cdots a_{J_2+1} a_J \cdots a_{J_3+1}\, d\, a_{J_1} \cdots a_1 a_{J_2} \cdots a_{J_1+1}) = \\
&\quad = \frac{1}{2}\left(\frac{N}{2}\right)^{J-J_3-1} \times \\
&\qquad \times (a_1 \cdots a_{J_1}\, a\, a_{J_1+1} \cdots a_{J_2} a_{J_2+1} a_{J_3}\, b\, d\, a_{J_1} \cdots a_1 a_{J_2} \cdots a_{J_1+1}\, c\, a_{J_3} \cdots a_{J_2+1}) \\
&\quad = \frac{1}{2}\left(\frac{N}{2}\right)^{J-J_3-1} (b\, d\, a_{J_1} \cdots a_1 a_{J_2} \cdots a_{J_1+1}\, c) \times \\
&\qquad \times (a_{J_3-1} \cdots a_{J_2+1} a_1 \cdots a_{J_1}\, a\, a_{J_1+1} \cdots a_{J_2} a_{J_2+1} \cdots a_{J_3-1}) \\
&\quad = \frac{1}{2}\left(\frac{N}{2}\right)^{J-J_2-2} (b\, d\, a_{J_1} \cdots a_1 a_{J_2} \cdots a_{J_1+1}\, c)(a_1 \cdots a_{J_1}\, a\, a_{J_1+1} \cdots a_{J_2})
\end{aligned}
$$

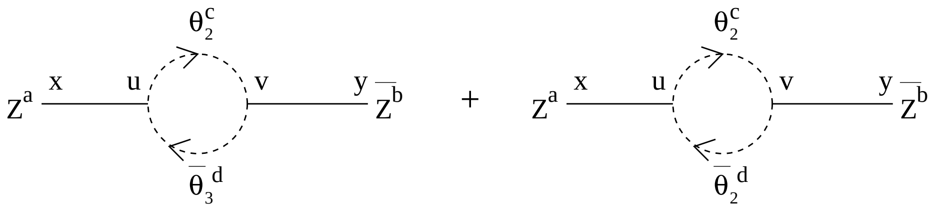


Figure 23: Chiral fermion loop contributions to self-energy of Z.

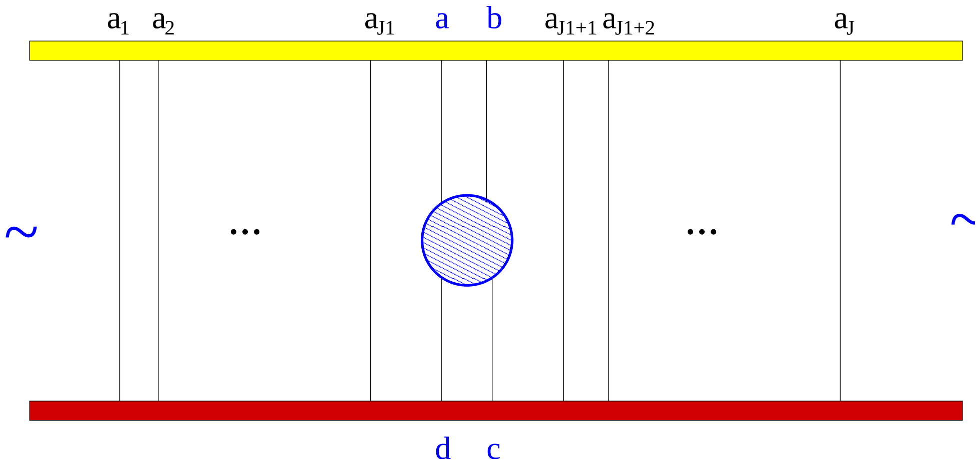


Figure 24: A planar diagram with same interaction vertex as figure 22.

$$= \frac{1}{2}\left(\frac{N}{2}\right)^{J-J_2+J_1-3}(a_{J_2}\cdots a_{J_1+1}cbdaa_{J_1+1}\cdots a_{J_2})$$
$$= \left(\frac{N}{2}\right)^{J-3}(cbda)\,.$$

Contraction with the vertex color, $f^{acp}f^{dbp}$ gives $(N/2)^{J+1}$.

Same interaction in a planar diagram, figure 24 would give,

$$(a_1\cdots a_{J_1}\,a\,b\,a_{J_1+1}\cdots a_J)(a_J\cdots a_{J_1+1}\,c\,d\,a_{J_1}\cdots a_1) = \frac{1}{2}\left(\frac{N}{2}\right)^{J-1}(abcd).$$

Contraction with the color factor of the vertex, $f^{adp}f^{bcp}$ gives, $(N/2)^{J+3}$ as the final color factor. Comparison of this result with the special graph result shows that special graph is indeed at torus level.

Let us move on to compute the F-term contribution at the planar level. Since we are interested in the anomalous dimension we consider equal momenta, $n = m$ for simplicity. Unlike in the case of D-terms there is a nice algebraic method to tackle with the calculation: Effective operator method [9]. We define the effective operator as the contraction of the vector operator

$$O^n_\mu = \partial^q_\mu(\phi Z^{J+1}) = (\partial_\mu\phi Z^{J+1}) + q\sum_{l=0}^{J} q^l(\phi Z^l\partial_\mu Z Z^{J-l})$$

with $f^{pab}\bar{Z}^a\phi^b$:

$$
\begin{aligned}
O_{\rm eff} \,=\, & -i\sum_{m=0}^{J}\left([a,p]Z^m a Z^{J-m}\right) G\partial_\mu G - \\
& -iq\sum_{l=0}^{J} q^l \sum_{m=0}^{l-1}\left([a,p]Z^m a Z^{l-m-1}\partial_\mu Z Z^{J-l}\right) G^2 - \\
& -iq\sum_{l=0}^{J} q^l \sum_{m=0}^{J-l-1}\left([a,p]Z^l a\partial_\mu Z Z^{J-l-m-1} Z^m\right) G^2 - \\
& -iq\sum_{l=0}^{J} q^l \left([a,p]Z^l a Z^{J-l}\right) G\partial_\mu G\,.
\end{aligned} \tag{B.2}
$$

Planar level contribution arises from the nearest-neighbour interactions, $m = 0, J$ in the first term, $m = 0$ in the second and third terms and $l = 0, J$ in the last term:

$$
O'^{\rm eff}_\mu = i(q-\bar{q})\left(\frac{N}{2}\right) G\partial_\mu G(pZ^J) + iq(q-1)\left(\frac{N}{2}\right) G^2 \sum_{l=0}^{J-1} q^l (pZ^l\partial_\mu Z Z^{J-l-1}) \tag{B.3}
$$

where we used (B.4) and $q^{J+2} = 1$. We obtain the two-point function as, $\langle O^n_\mu \bar{O}^n_\nu\rangle = \langle O^{\rm eff}_\mu \bar{O}^{\rm eff}_\nu\rangle$. To compute various terms we will need additional contraction identities,

$$
\begin{aligned}
{\rm Tr}(Z^a\bar{Z}^a) \,&=\, N^{a+1} + \mathcal{O}(N^{a-1}) \\
{\rm Tr}(Z^a)\,{\rm Tr}(\bar{Z}^a) \,&=\, aN^a + \mathcal{O}(N^{a-2})
\end{aligned} \tag{B.4}
$$

which are derived by counting the number of ways one may perform the Wick contractions within each trace structure while obtaining a maximal power of N. Leading order terms show the planar level contributions.

Among the four pieces in $\langle O^{\rm eff}_\mu \bar{O}^{\rm eff}_\nu\rangle$ the term arising from contraction of second term in $O^{\rm eff}_\mu$ with second term in $\bar{O}^{\rm eff}_\nu$ is the easiest to evaluate. One gets,

$$
-\,(q-\bar{q})^2\partial_\mu G\partial_\nu G G^J\left(\frac{N}{2}\right)^{J+3}. \tag{B.5}
$$

Contraction of first term in $O^{\rm eff}_\mu$ with second term in $\bar{O}^{\rm eff}_\nu$ gives,

$$
\begin{aligned}
-\left(\frac{N}{2}\right)^2 & (q-\bar{q})q(q-1)G\partial_\mu G\times \\
\times\sum_{l=0}^{J-1} q^l (Z^l\partial_\nu Z Z^{J-l-1}p)(p\bar{Z}^J) \,&=\, -(N/2)^{J+3}(q-\bar{q})q(q-1)G\partial_\mu\partial_\nu\left(\sum_{l=0}^{J-1} q^l\right) \\
&=\, (q-\bar{q})^2\partial_\mu G\partial_\nu G G^J\left(\frac{N}{2}\right)^{J+3}.
\end{aligned} \tag{B.6}
$$

The other cross-term yields the same expression hence doubles (B.6). Contraction of first terms of $O^{\rm eff}_\mu$ and $\bar{O}^{\rm eff}_\nu$ is,

$$
G^4\frac{1}{2}\left(\frac{N}{2}\right)^2(1-q)(1-\bar{q})\sum_{l,l'=0}^{J-1} q^l\bar{q}^{l'}(Z^l\partial_\mu Z Z^{J-l-1}\bar{Z}^{J-l'-1}\partial_\nu\bar{Z}\bar{Z}^{l'})\,.
$$

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One can break up the trace into two pieces by contracting $\partial_\mu Z$ with a $\bar{Z}$ in the first group, with $\partial_\nu \bar{Z}$ or with a $\bar{Z}$ in the last group. First possibility gives, (up to the factors in front of the sum)

$$\frac{1}{2}\partial_\mu G \sum_{p=0}^{J-l'-2} (Z^{J-l-1}\bar{Z}^p)(\bar{Z}^{J-l'-p-2}\partial_\nu \bar{Z}\bar{Z}^{l'}Z^l) =$$

$$= \frac{1}{4}\partial_\mu G \partial_\nu G \sum_{p=0}^{J-l'-2}\sum_{r=0}^{l-1}(Z^{J-l-1}\bar{Z}^p)(Z^r\bar{Z}^{l'})(Z^{l-r-1}\bar{Z}^{J-l'-p-2})$$

$$= \left(\frac{N}{2}\right)^{J-2}\frac{N^3}{4}G^{J-2}\partial_\mu G\partial_\nu G\sum_{l<l'} q^l \bar{q}^{l'}\,.$$

Including the factors in front, one has,

$$(1-q)(1-\bar{q})\left(\frac{N}{2}\right)^{J+3} G^J \partial_\mu G \partial_\nu G \sum_{l<l'} q^l\bar{q}^{l'}\,.$$

whereas the second possibility gives,

$$(1-q)(1-\bar{q})\left(\frac{N}{2}\right)^{J+3} G^{J+1}\partial_\mu\partial_\nu G\,.$$

A similar calculation shows that third possibility yields,

$$(1-q)(1-\bar{q})\left(\frac{N}{2}\right)^{J+3} G^J \partial_\mu G \partial_\nu G \sum_{l>l'} q^l\bar{q}^{l'}\,,$$

giving all in all,

$$\left(\frac{N}{2}\right)^{J+3} G^{J+2}\left\{J\frac{2J_{\mu\nu}}{(x-y)^2} + (1+q)(1+\bar{q})\partial_\mu G\partial_\nu G\right\}. \tag{B.7}$$

Combining the various pieces we have computed, we see that (B.5), (B.6) and the second term in (B.7) cancels out, leaving us with,

$$\left(\frac{N}{2}\right)^{J+3} G^{J+2} J\frac{2J_{\mu\nu}}{(x-y)^2}\,. \tag{B.8}$$

Taking into account the integral over the interaction vertex, (5.6), one arrives at the final contribution of the F-terms,

$$\langle O^n_\mu(x)\overline{O}^m_\nu(y)\rangle_F = -\lambda' n^2 \ln\left((x-y)^2\Lambda^2\right)\frac{J_{\mu\nu}(x,y)}{(x-y)^2}G(x,y)^{J+2} \tag{B.9}$$

which exactly equals the sum of the contributions from D-terms, self-energy and external gluons. Therefore total anomalous dimension is twice the dimension in B.9).

One can use the effective operator method also to calculate the F-term contribution to torus anomalous dimension. For that purpose one should keep the second order terms in the expansion of (B.4). This calculation was done in appendix D of [9] and one gets the same expression as (5.7).

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